MECHANICS.

MECHANICS

REQUIRED FOR THE ADDITIONAL SUBJECTS FOR HONOURS AT THE PREVIOUS EXAMINATION

AND FOR THE ORDINARY DEGREE.

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CAMBRIDGE:
DEIGHTON, BELL, AND CO.
LONDON: BELL AND DALDY.
1867.

Cambridge:
PRINTED BY JONATHAN PALMER.

PREFACE.

THE present work is one of a series intended for the Previous Examination and the Ordinary Degree, according to the new scheme which came into operation last December.

I have endeavoured to make the definitions and investigations clear, precise, and accurate; and to present them in a convenient form for writing out at an examination.

In Article 15 I have given a proof of the first part of the Parallelogram of Forces, which, by its brevity and simplicity, will, I trust, free Duchayla's proof from the reproach of being "long and difficult."

In Article 44 the conditions of equilibrium of a rigid body acted on by any number of forces in the same plane are deduced without the aid of analysis. I have not found these conditions satisfactorily deduced in any elementary work without the aid of analysis or the composition of couples. Trigonometrical methods have been very sparingly introduced, and wherever they occur, the Article or Section has been marked with an asterisk.

Though there is nothing in the work which may not fairly be set from time to time at the Previous or Degree Examination, yet as all students do not require to read exactly the same amount for a pass examination, I have left the more precise indication of the course for individual cases to the judgment of the college and private tutors.

The text is illustrated by a large number of Examples and Problems, chiefly numerical, most of which have been constructed expressly for this work. These examples have been carefully worked out, and the results, with occasional hints for solution, are given at the end of the book.

My best thanks are due to several friends for valuable advice and assistance during the progress of the present work.

I shall be thankful for any corrections or suggestions from teachers or students.

J. M^c^DOWELL.

PEMBROKE COLLEGE,
CAMBRIDGE,
February 14th, 1867.

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ELEMENTARY STATICS.

CHAPTER I.

DEFINITIONS AND EXPLANATIONS.

1. *Matter* is whatever affects our senses in any way whatever.

A body is a portion of matter limited in every direction, and consequently it has length, breadth, and thickness, or *three dimensions.*

A material particle, or simply *a particle*, is a body indefinitely small in all its dimensions.

A particle is often called a *material point*, or a *point*, and in all investigations involving its position, it is treated as a *Geometrical point.*

Every body may be considered as composed of an indefinite number of *particles* or *molecules.* The *mass* of a body is the quantity of matter which it contains. It is shewn in Works on Dynamics that the mass of every body is proportional to its weight.

A body is said to be *at rest* when its particles continually occupy the same position in space.

A body is said to be *in motion* when its particles continuously occupy different positions in space at successive instants of time.

Force is any cause which produces, or tends to produce, any change in the state of rest or motion of a particle or of a body.

The *Science of Mechanics* consists of two parts, *Statics* and *Dynamics*. *Statics* treats of the effects of forces on a particle, a body or a system of bodies, at rest.

Dynamics treats of the effects of forces on a particle, a body, or a system of bodies, in motion.

2. *A rigid body* or *a rigid system of bodies* is one which preserves an invariable form under the action of any finite forces.

When a system of forces acting on a particle, or system of particles connected together, produces no change in the state of the particle or system of particles, the forces are said to be *in equilibrium*.

A particle or a system of particles *at rest* is also said to be *in equilibrium* under the action of such forces.

3. It is obvious that if a system of forces in equilibrium be applied to a particle or to a rigid system, or suppressed, no change will take place in the condition of the particle or of the rigid system.

If one body press against another at any point, the mutual actions of the bodies are equal and opposite, and the forces exerted by the bodies on one another are called *pressures*.

The equality of the mutual actions between two bodies connected in any way, is often summed up in the statement that *action and reaction are equal and opposite*.

If a body be pulled by a string, the force exerted on it by the string is called *tension*. If the string be *assumed* to be perfectly flexible and without weight, the tension will be the same throughout its length.

4. If a force can support a weight of 1 lb., 2 lbs., 3 lbs., &c., it is called a force of 1 lb., 2 lbs., 3 lbs., &c.; and generally, if a force can support a weight of P lbs., it is called a force of P lbs.

Here P is the *numerical representative* of the force, 1 lb. being the unit of force. Any other unit may be chosen according to convenience.

5. Hence, forces which can support the same weight are equal. We may also define equal forces as follows.

Two forces are *equal* when, being applied to the same point in the same straight line, but in opposite directions, they constitute an equilibrium. If two of these equal forces act in the same direction, we get a force double of either; if three of them act in the same direction, we get a force three times each of them, and so on. If we take one of these equal forces as the unit of force, then P will be the *numerical representative* of a force which contains P units of force or P of those equal forces.

6. The particle or point upon which a force acts is called the *point of application* of the force. If a force act upon a particle, the direction in which it would cause the particle, if free, *to begin* to move, is called the *line of action* of the force.

Any straight line parallel to the line of action of a force is said to be *the direction of the force.*

7. Three things are therefore to be considered in a force; namely, its *point of application*, its *line of action*, or its *direction*, and its *intensity* or *magnitude*. When these are known the force is completely determined.

8. It is a fact suggested by experiment and confirmed by induction, that *the point of application of a force may be transferred to any other point in its line of action; provided this latter point be rigidly connected with the former point.* This is called *the Principle of the Transmission of Force.*

9. *Forces can be represented Geometrically by straight lines.*

For a straight line can be drawn from the point of application of a force, in the direction of the force, and proportional to its magnitude.

Thus, if $P = 5$ lbs. be a given force acting at the point A, in the straight line AB, and if AC be taken as the unit of length and AB made equal to $5AC$; then AB will represent P, for AB is drawn from the point of application A, in the direction AB of the force, and contains as many units of length as the force contains units of force.

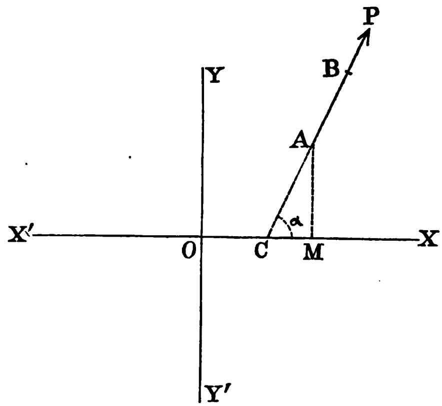
If AB represent a force acting in the direction indicated by the arrow, then BA will represent an equal force acting in the opposite direction.

10. *Forces can be represented Algebraically.*

Let P be a given force acting at A in the straight line AB.

Let $X'X$, YY' be two fixed straight lines intersecting at right angles in O, and let AB the direction of the given force P meet $X'X$ in C.

Draw AM perpendicular to $X'X$ and let $OM = a$, $AM = b$, and the angle $ACX = \alpha$.



If a, b, α, P be given, it is clear that the force is completely determined; for a, b, determine its point of application A, α determines its direction and P its magnitude.

If a force in any assigned direction be considered *positive*, a force in the opposite direction will be *negative*, the same conventions respecting *positive* and *negative*, or the use of the signs + and –, being adopted, as in the TRIGONOMETRY, Art. 2.

* OM is usually called the *abscissa* of the point A, and AM the *ordinate*. OM and AM taken together are called the *co-ordinates* of A. $X'X$ is called the *axis of* x, and YY' the *axis of* y, these two lines together being called the *co-ordinate axes* or the *axes of co-ordinates*.

CHAPTER II.

ON FORCES ACTING AT A POINT.

11. *Resultant.*

When a single force can equilibrate any number of forces applied to a system of points rigidly connected together, these forces can be replaced by a single force R, equal and opposite to the first. The force R is called the ***resultant*** of the forces which it replaces, and these replaced forces are called its *components.*

From this definition it follows that if a system of forces be in equilibrium, any one of them is equal and opposite to the resultant of all the rest.

We may also infer from this definition that if any number of forces act upon a particle, their resultant is in the direction in which the particle (if subjected only to the action of these forces) would *begin to move:* hence, if two forces act upon a point, the line of action of their resultant lies *within* the angle ($< 180°$) contained by the lines of action of the two component forces.

12. *Composition of Forces acting in the same straight line.*

The resultant of any number of forces which act in the same straight line is equal to the excess of the sum of those which act in one direction over the sum of those which act in the opposite direction, and this resultant acts in the direction of the forces which have the greater sum.

In other words, the *magnitude* and *direction* of the resultant are given by the *Algebraical sum* of the forces, by regarding as positive those which act in one direction, and as negative those which act in the opposite direction.

13. LEMMAS.

(1) *The diagonals of a rhombus bisect its angles.*

(2) *The diagonals of a rhombus bisect one another at right angles.* (McDowell's EXERCISES ON EUCLID, p. 13, Cor.)

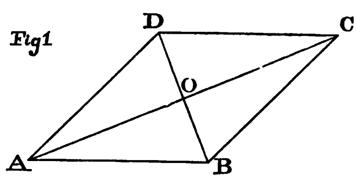
(3) *The diagonals of any parallelogram bisect one another.* (EXERCISES, p. 9, No. 11.)

(4) *The bisectors of the sides of a triangle drawn from the opposite angles pass through the same point and cut each other in a point of trisection.* (EXERCISES, p. 3, No. 5.)

Also the straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to half of it.

(1) Let $ABCD$ be a rhombus.

Fig 1



In the triangles ABC, ADC, the sides AB, AC are equal to AD, AC each to each, and the base BC is equal to the base DC, therefore (Euc. I. 8) the angle BAC is equal to the angle DAC, and therefore the diagonal AC bisects the angle BAD.

(2) Because in the triangles BAO and DAO, the sides BA, AO are equal to DA, AO, and the angle BAO equal to the angle DAO, therefore the base BO is equal to the

base DO, and the angle AOB to the angle AOD, and therefore the angles AOB and AOD are right angles.

(3) Suppose $ABCD$ to be any parallelogram.

Because AB is parallel to DC, therefore the alternate angles BAO and DCO, ABO and CDO are equal.

Also $AB = DC$; therefore (Euclid I. 26) $AO = OC$ and $BO = OD$. Q. E. D.

(4) Let D, E, F be the middle points of the sides of the triangle ABC.

Because $AF : FB :: AE : EC$, therefore (Euclid VI. 2) FE is parallel to BC, and the triangles AFE and ABC are equiangular,

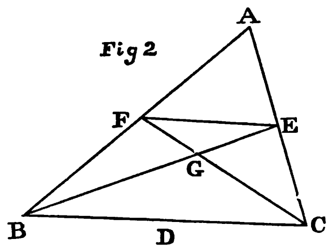


Fig 2

therefore (Euc. VI. 4) $AF : FE :: AB : BC$,

or alternately,

$$AF : AB :: FE : BC;$$

but $\quad AF = \frac{1}{2}AB, \quad \therefore \; FE = \frac{1}{2}BC.$

Also the triangles FGE and BGC are equiangular,

therefore (Euc. VI. 4) $FE : EG :: CB : BG$,

or alternately

$$FE : CB :: EG : BG,$$

but $\quad FE = \frac{1}{2}CB, \quad \therefore \; EG = \frac{1}{2}BG,$

and therefore $\quad EG = \frac{1}{3}BE.$

In the same way it can be shewn that if AD be drawn, it will cut off from BE a third part towards AC. Therefore the three bisectors of the sides of a triangle pass through the same point, and cut each other in a point of trisection.

Q. E. D.

14. THE PARALLELOGRAM OF FORCES.

If two forces acting upon a point be represented in direction and magnitude by two straight lines drawn from that point, and if a parallelogram be constructed with these two lines for adjacent sides, then the diagonal of this parallelogram, which passes through the point, will represent the resultant of the two forces in direction and magnitude.

In the following proof of this proposition all the points of the system are supposed rigidly connected together, and therefore a force may be applied at any point in its line of action without altering its effect. This proposition may be divided into three parts, namely:—

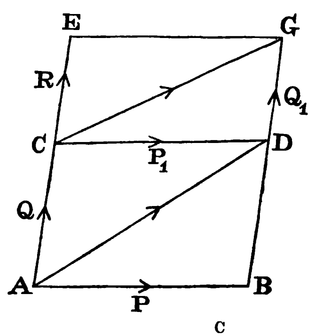
(1) The diagonal represents the direction of the resultant when the forces are *commensurable.*

(2) The diagonal represents the direction of the resultant when the forces are *incommensurable.*

(3) The diagonal represents the magnitude of the resultant.

(1) is true when the forces are *equal;* for it is clear that, in this case, the direction of the resultant *bisects* the angle between the directions of the components, since there is no reason why it should be nearer to the direction of one component than to that of the other, and by Art. 13 (1) the diagonals of a rhombus bisect its angles, and therefore when the forces are *equal* (1) is true.

(4) If (1) be true for any two sets of forces P and Q, P and R, it will be true for P and $Q+R$.



Let A be the point of application of these forces, and let P be represented by AB, Q by AC, and transfer the point of application of R to C, and let R be represented by CE. Complete the parallelograms BC and DE.

C

By the hypothesis (4), the resultant of P and Q acts along AD.

Let this resultant be applied at D, and there be replaced by the two forces P_1 and Q_1 respectively equal to P and Q, and acting along CD and DG parallel to AB and AC.

Let now P_1 have its point of application transferred to C, and Q_1 to G.

R and P_1 acting at C, have by (4) a resultant acting along CG. Let this resultant have its point of application transferred to G.

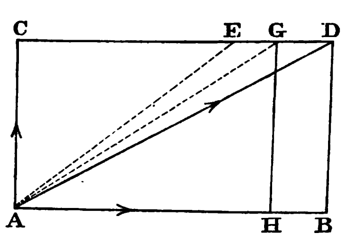
We have thus transferred the point of application of all the forces to G, without altering the effect of the forces. Hence G is a point on their resultant, but A is also a point on their resultant, therefore AG is the direction of the resultant in any case in which the hypothesis (4) holds true; but it has been proved to be true when P, Q, and R are each equal to the same force F.

Therefore it holds for F and $2F$, and therefore for F and $3F$, and so on; therefore it holds for F and $m.F$ where m is any positive integer.

Again, putting $P=m.F$, $Q=R=F$, it holds for $m.F$ and $2F$, and therefore for $m.F$ and $3F$, and so on; therefore it holds for $m.F$ and $n.F$ where m and n are any positive integers.

Now any two *commensurable* forces may be represented by $m.F$, and $n.F$, where F is their common measure, and m and n the number of times which they contain F. Hence (1) is completely proved.

Next let AB, AC represent two *incommensurable* forces.



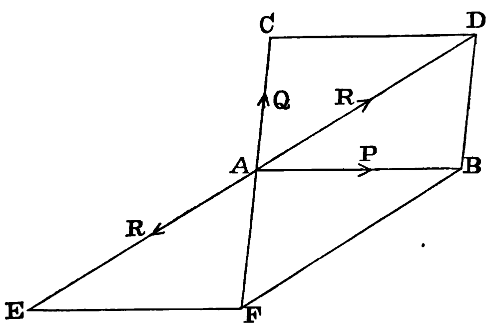
Complete the parallelogram BC.

If AD be not the direction of the resultant, let, if possible, some other line AE be the direction.

Divide AC into a number of equal parts, each less than ED, and mark off from CD, beginning at C, as many parts as possible, each equal to one of these equal parts. The last division G will clearly fall between E and D. Draw GH parallel to AC. Since AH and AC represent commensurable forces, their resultant will act along AG. Therefore the two forces represented by AB and AC may be replaced by a force acting in AG and a force represented by HB acting at A, and these two forces will have a resultant acting *within* the angle BAG, that is, the resultant of the two given forces lies *within* the angle BAG; but by hypothesis, it acts in AE *without* the angle BAG, which is absurd.

In the same way it may be proved that no line but AD is the direction of the resultant of the forces represented by AB and AC. Therefore (1) and (2) are completely proved.

[When a force P is represented by a straight line AB, instead of saying "the force represented by AB", we shall often use the shorter expression, *the force AB*. When forces are represented by straight lines we may also speak of them being parallel or intersecting.]



Lastly, let AB, AC represent any two forces P and Q, then shall the diagonal AD of the parallelogram $ABDC$ represent the resultant in magnitude.

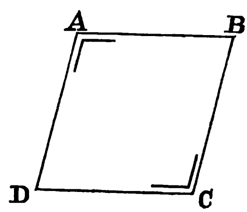
Suppose a force AE in the production of DA to be equal to the unknown resultant of P and Q, and complete the parallelogram BE.

Because the three forces AB, AC, AE are in equilibrium, therefore (Art. 11) AC is equal and opposite to the resultant of AB and AE; but it has been proved that AF is the direction of this resultant, therefore CAF is a straight line, and therefore DF is a parallelogram; therefore $AD = BF = AE$.

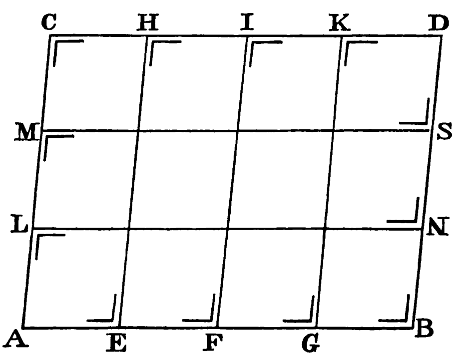
Therefore AD represents the resultant of AB, AC in magnitude, and (3) is therefore true.

Hence "the Parallelogram of Forces" has been completely proved.

15. M. Sturm has proved the first part of "the Parallelogram of Forces" in the following manner.



Let $ABCD$ be a rhombus of invariable form. Apply at the points A and C four equal forces acting along the sides AB, AD, CB, CD. It may be regarded as evident that the equal forces applied at A give a resultant acting along the bisector AC of the angle BAD, and that the forces applied at C give a resultant equal and directly opposite to the first. The system therefore remains in equilibrium under the action of the four forces.



Let now f be a common measure of the two forces P and Q represented by AB, AC respectively, and let us suppose, for the purpose of fixing the ideas, that $P=4f$, $Q=3f$. Divide AB into four equal parts, and AC into three parts equal to each other, and consequently equal to the first. Draw EH, FI, GK, parallel to AC and MS, LN parallel to AB.

We shall not disturb the state of the system by applying to the vertices L and E, M and F, C and G, H and B, I and N, K and S, of the rhombuses LE, MF, CG, HB, IN, KS, and in the direction of the sides of these rhombuses, forces equal to f. But the equal and opposite forces applied at the extremities of the straight lines HE, IF, KG, MS, LN destroy each other. Therefore there remain only 4 forces $=f$ acting along CD, and 3 forces equal to f acting along BD. The four first compound into a single force $=P$, which we may suppose applied at the point D, and likewise the three others give a resultant $=Q$, and applied at the same point D. It results from this that the system of the two forces P and Q applied at A can be replaced by two forces P and Q applied at the point D. Therefore the resultant passes through the point D, but it already passes through the point A; therefore it acts along AD. Q. E. D.

When we find the resultant of two or more forces, we are said to *compound* the forces; and conversely, when we find several forces equivalent to a single force, we are said to *resolve* the single force into the others. The two processes are called the *composition* and *resolution* of forces.

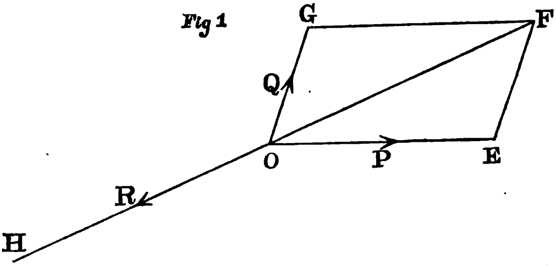
16. THE TRIANGLE OF FORCES.

(1) *If three forces acting at a point be in equilibrium, and if any triangle be constructed having its sides parallel to the directions of the forces, the sides of the triangle will be proportional to the forces.* (This is also true when one force is the resultant of the other two.)

And conversely,

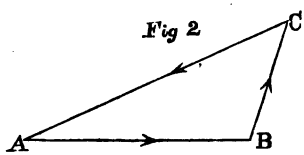
(2) *If three forces acting at a point be represented in direction and magnitude by the sides of a triangle* taken in order, *they will be in equilibrium.*

Fig 1



(1) Let the forces P, Q, R acting at the point O be in equilibrium, and be represented by OE, OG, OH; construct the triangle ABC having its sides parallel to the directions of P, Q, R; then shall

Fig 2



$$AB : BC : CA :: P : Q : R.$$

Complete the parallelogram EG. Because P, Q, R are in equilibrium, OF and OH are equal and in the same straight line. Therefore the sides of the triangles OEF, ABC are respectively parallel, and therefore

$$OE : EF : FO :: AB : BC : CA,$$

that is, $P : Q : R :: AB : BC : CA.$ Q. E. D.

(2) Let AB, BC, CA the sides of the triangle ABC taken in order represent in direction and magnitude the three forces P, Q, R, which act at the point O; then shall P, Q, R be in equilibrium.

Take OE, OG, OH to represent the three forces, and therefore equal and parallel to the sides of the triangle ABC taken in order.

Complete the parallelogram EG; therefore OE, EF are equal and parallel to AB, BC, each to each, and therefore OF is equal and parallel to AC; therefore OF the resultant of P and Q is equal and opposite to R.

Therefore P, Q, R are in equilibrium. Q. E. D.

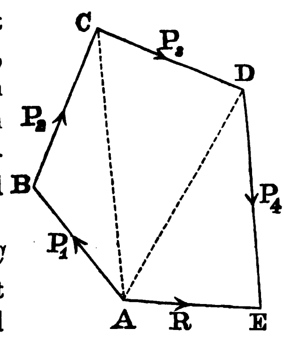
17. *If two sides of a triangle* taken in order *from an angular point represent in direction and magnitude two forces acting at that point, then the third side* acting from the point *will represent the resultant in direction and magnitude.*

(See fig. 2, Art. 16.)

For, by the Triangle of Forces, the three forces AB, BC, CA acting at A are in equilibrium; therefore AC is the resultant of AB, BC. Q. E. D.

18. *To find* (geometrically or graphically) *the resultant of any number of forces acting at a point.*

Let the forces P_1, P_2, P_3, P_4 act at the point A. Draw AB, BC, CD, DE to represent the forces in direction and magnitude, and join AE. AE will represent the required resultant (R) in direction and magnitude.



By Art. (17) the forces AB, BC may be replaced by their resultant AC, then AC, CD may be replaced by their resultant AD, that is, the forces AB, BC, CD may be replaced by their resultant AD, and lastly, AD, DE may be replaced by their resultant AE; therefore AE represents the resultant of the given forces in direction and magnitude.

Q. E. F.

19. THE POLYGON OF FORCES.

If the sides of a polygon, taken in order, *represent in direction and magnitude any number of forces acting at a point, these forces will be in equilibrium.*

(See fig. Art. 18.)

Let AB, BC, CD, DE, EA represent in direction and magnitude a system of forces acting at A; then shall these forces be in equilibrium.

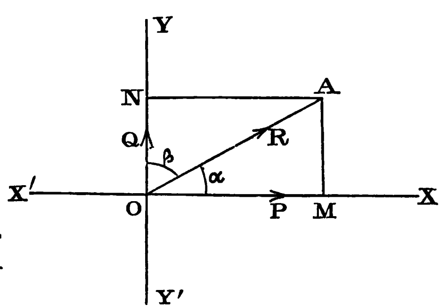
It is shewn in Art. 18, that AE represents the resultant of all these forces except EA.

The system of forces is thus reduced to AE and EA, which are clearly in equilibrium. Q. E. D.

N.B.—If this proposition were set at an examination, the part of the proof contained in Art. 18 should be included. This remark will sometimes apply to other propositions.

20. (See fig. 1, Art. 13.) If $ABCD$ be a parallelogram, then, by Art. 13 (3), $AO = OC$ and $BO = OD$. Hence if AB, AD represent two forces acting at the point A, $2AO$, that is, *twice the bisector of the base BD of the triangle ABD will represent the resultant.*

21. *To resolve a given force in two given directions.*



Let OA represent the given force R, OX, OY the given directions of the components.

Draw AM parallel to OY, and AN parallel to OX; then, by the Parallelogram of Forces, $OM (= P)$, $ON (= Q)$ represent the required components.

*22. (Fig. Art. 21.) *To find the resultant* R *in terms of the components* P *and* Q, *and to find* P *and* Q *in terms of* R.

Let $YOX=\omega$, and let the direction of R make angles α, β with the directions of P and Q respectively.

In the triangle AOM, $MA=Q$, and the angle $AMO = \pi-\omega$.

Therefore (TRIG. Art. 54)

$$AO^2=MA^2+OM^2-2MA.OM\cos AMO,$$

or $$R^2=P^2+Q^2-2P.Q\cos(\pi-\omega);$$

$\therefore$ $$R^2=P^2+Q^2+2P.Q\cos\omega.$$

Also $$\frac{OM}{OA}=\frac{\sin OAM}{\sin AMO},$$

that is, $$\frac{P}{R}=\frac{\sin\beta}{\sin(\pi-\omega)}=\frac{\sin\beta}{\sin\omega}; \quad \therefore\ P=R.\frac{\sin\beta}{\sin\omega};$$

and $$\frac{MA}{OA}=\frac{\sin MOA}{\sin AMO},$$

that is, $$\frac{Q}{R}=\frac{\sin\alpha}{\sin(\pi-\omega)}=\frac{\sin\alpha}{\sin\omega}; \quad \therefore\ Q=R.\frac{\sin\alpha}{\sin\omega}.$$

If the angle $\omega=\frac{\pi}{2}$ we have $\left(\text{since } \sin\frac{\pi}{2}=1\right)$ from the above values of P and Q, or directly from the figure,

$$P=R\cos\alpha, \text{ and } Q=R\cos\beta=R\sin\alpha.$$

$R\cos\alpha$ is called the *resolved part* of R in the direction making an angle α with the direction of R.

Hence, *To find the resolved part of a force in a given direction, multiply the force by the cosine of the angle between its direction and the given direction.*

23. (Fig. Art. 21.) *When the directions of* P *and* Q, *the components of* R, *are inclined at a right angle, shew that* $R=\sqrt{(P^2+Q^2)}$.

Here the triangle AOM has the angle AMO right; therefore (Euc. I. 47)

$$AO^2 = OM^2 + MA^2,$$

that is, $$R^2 = P^2 + Q^2;$$

therefore $$R = \sqrt{(P^2 + Q^2)}.$$ Q. E. D.

*24. *If three forces acting at a point be in equilibrium, each force is proportional to the sine of the angle between the other two.*

(Fig. 1. Art. 16.) Let the three forces P, Q, R, acting at O, be in equilibrium, and be represented by OE, OG, OH, and complete the parallelogram EG.

From the triangle OEF we have

$$\frac{OE}{\sin OFE} = \frac{EF}{\sin EOF} = \frac{FO}{\sin FEO} \text{ or } \frac{P}{\sin FOG} = \frac{Q}{\sin EOF} = \frac{R}{\sin FEO};$$

but $\sin FOG = \sin GOH$, since these angles are supplemental,

$\sin EOF = \sin HOE$, ...

$\sin FEO = \sin EOG$, ...

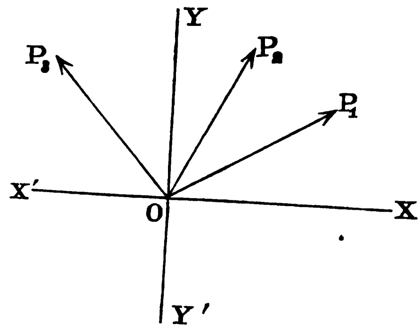
therefore $$\frac{P}{\sin GOH} = \frac{Q}{\sin HOE} = \frac{R}{\sin EOG};$$

or, if we denote the angle between the directions of two forces P and Q by (P, Q),

$$\frac{P}{\sin(Q, R)} = \frac{Q}{\sin(R, P)} = \frac{R}{\sin(P, Q)}.$$ Q. E. D.

*25. *To find expressions for the direction and magnitude of the resultant of any number of forces acting in one plane at a point.*

Let the forces P_1, P_2, P_3, &c. act in one plane at the point O. Through O draw $X'X$, YY' at right angles to one



another in the plane of the forces, and let the directions of P_1, P_2 &c. make angles α_1, α_2 &c. with OX.

By Art. 22 the resolved parts of P_1, P_2, &c. are $P_1 \cos\alpha_1$, $P_2 \cos\alpha_2$ &c. in the direction OX, and $P_1 \sin\alpha_1$, $P_2 \sin\alpha_2$ &c. in the direction OY.

Put X for $P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + \&c.$,

and Y for $P_1 \sin\alpha_1 + P_2 \sin\alpha_2 + \&c.$,

and let R be the required resultant making an angle θ with OX; then we have

$$R\cos\theta = X,$$

$$R\sin\theta = Y;$$

$$\therefore \frac{R\sin\theta}{R\cos\theta} = \frac{Y}{X}, \text{ or } \tan\theta = \frac{Y}{X} = \frac{P_1 \sin\alpha_1 + P_2 \sin\alpha_2 + \&c.}{P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + \&c.},$$

which determines the direction of the resultant.

Also $(R\cos\theta)^2 + (R\sin\theta)^2$ or $R^2 = X^2 + Y^2$;

therefore $$R = \sqrt{(X^2 + Y^2)},$$

which determines the magnitude of the resultant.

*26. *To find the conditions of equilibrium when any number of forces act in one plane at a point.*

By the last Article we have always $R^2 = X^2 + Y^2$; but for equilibrium we must have $R = 0$,

and therefore $X = 0$ and $Y = 0$,

or $P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0$, and $P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \&c. = 0$;

that is, when the forces are in equilibrium, *the sums of the resolved parts of the forces in any two directions at right angles to one another must be separately zero.*

CHAPTER III.

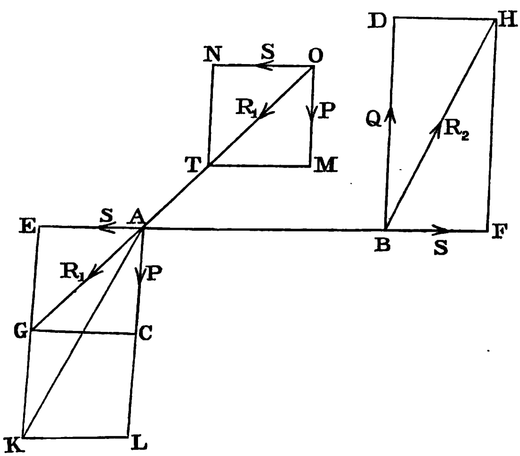
ON PARALLEL FORCES.

27. LEMMAS.

(1) Let P and Q be two parallel forces acting in opposite directions at the points A and B, and S, S two equal forces also acting at A and B in opposite directions, but along the same straight line; then *the line of action of* R_1, *the resultant of* P *and* S, *will meet the line of action of* R_2, *the resultant of* Q *and* S, *on the side of the* greater *force* Q.

(2) *Also any force, as* R_1, *can be resolved into two forces equal and parallel to its original components, and applied at any point in its line of action.*

Let P and S at A be represented by AC and AE, Q and



S at B by BD and BF. Make $AL = BD$ and complete the parallelograms *CE*, *EL*, *DF*. Also produce *GA* to any point *O*, make $OT = AG$, and draw *OM*, *TN* parallel to *AC*, and *ON*, *TM* parallel to *AE*.

The triangles *AKL*, *HBF* are clearly equal in all respects, and *AL* is parallel to *HF*; therefore *KA* is parallel to *BH*, and therefore *GA* and *BH* will meet *above* *AB* and to the *right* of *BD*; therefore (1) is true.

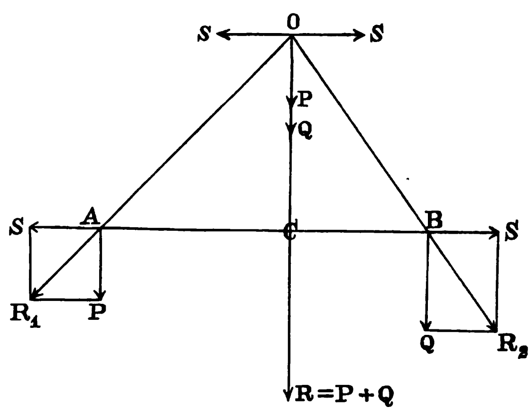
It is clear, from the construction, that the parallelograms *CE* and *MN* are equal in all respects. Therefore (2) is true.

Q. E. D.

[In writing out the next two propositions, these Lemmas need not be included, unless expressly asked.]

28. *To find the resultant of two parallel forces which act at any two points of a rigid system, in the same direction* (but not in the same straight line).

Let the two forces *P* and *Q* act at the points *A* and *B*, and in the parallel directions *AP*, *BQ*.



At *A* and *B* apply two equal and opposite forces *S*, *S* acting in the straight line AB. This will not disturb the state of the system.

Let R_1 be the resultant of P and S acting at A, and R_2 the resultant of Q and S acting at B.

Let the lines of action of R_1 and R_2 meet in O, and the points of application of these forces be transferred to O, and let R_1, R_2 acting at O be resolved [Art. 27 (2)] into forces equal and parallel to their original components.

The pair of equal and opposite forces S, S acting at O destroy one another, and there remain only the two forces P and Q acting along OC, which is parallel to AP. Therefore if R be the required resultant, we have

$$R = P + Q \quad \text{.......................} \quad (1).$$

R may be supposed to act at C.

Again, the sides of the triangle ACO are respectively parallel to the three forces P, S, R_1 acting at A, and the sides of the triangle BCO are respectively parallel to the three forces Q, S, R_2 acting at B, and therefore, by the Triangle of Forces (see Arts. 16, 17),

$$\frac{P}{S} = \frac{OC}{CA} \text{ and } \frac{S}{Q} = \frac{CB}{OC};$$

$$\therefore \frac{P}{S} \cdot \frac{S}{Q} = \frac{OC}{CA} \cdot \frac{CB}{OC} \text{ or } \frac{P}{Q} = \frac{CB}{CA} \text{.................} (2),$$

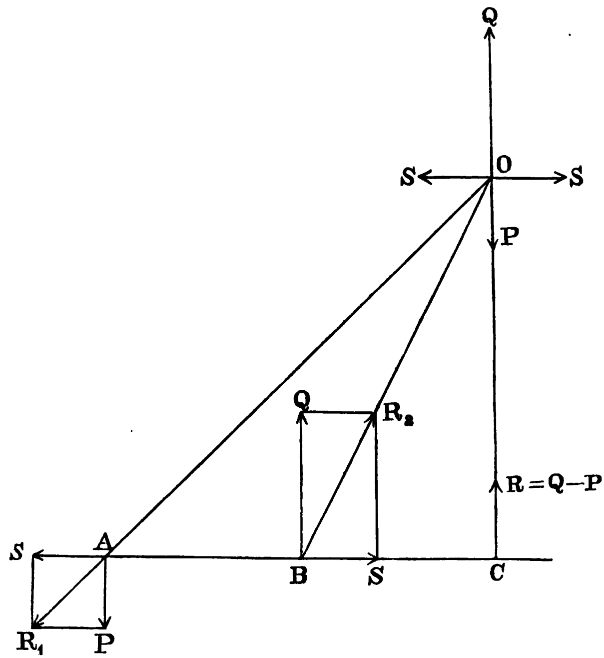
that is, $P : Q :: CB : CA$, or $P . CA = Q . CB$.

Hence the two forces P *and* Q *are inversely as the distances of their points of application from the point of application of the resultant; the resultant is equal to their sum, and acts in the same direction.*

29. *To find the resultant of two parallel forces which act at any two points of a rigid system, in opposite directions* (but not in the same straight line).

Let the two forces P and Q act at the points A and B, in the parallel directions AP, BQ, and suppose Q greater than P.

At A and B apply two equal and opposite forces S, S acting in the straight line AB. This will not disturb the state of the system.



Let R_1 be the resultant of P and S acting at A, and R_2 the resultant of Q and S acting at B. Let the lines of action of R_1 and R_2 meet in O, and let the points of application of these forces be transferred to O. Let R_1, R_2 acting at O be resolved [Art. 27 (2)] into forces equal and parallel to their original components.

The pair of equal and opposite forces S, S acting at O

destroy one another, and there remain only the force P acting in the direction OC, which is parallel to AP, and Q acting in the opposite direction; therefore, if R be the required resultant, we have

$$R = Q - P \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

R acts in the same direction as the greater force Q, and may be supposed to act at C.

Again, the sides of the triangle ACO are respectively parallel to the directions of the three forces P, S, R_1 acting at A, and the sides of the triangle BCO are respectively parallel to the directions of the three forces Q, S, R_2 acting at B; and therefore, by the Triangle of Forces (see Arts. 16, 17),

$$\frac{P}{S} = \frac{OC}{CA} \text{ and } \frac{S}{Q} = \frac{CB}{OC};$$

$$\therefore \quad \frac{P}{S} \cdot \frac{S}{Q} = \frac{OC}{CA} \cdot \frac{CB}{OC},$$

or

$$\frac{P}{Q} = \frac{CB}{CA} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

that is, $P:Q::CB:CA$, or $P.CA = Q.CB$.

Hence, *the two forces* P *and* Q *are inversely as the distances of their points of application from the point of application of the resultant; the resultant is equal to the difference of the two forces, acts in the direction of the greater force, and on the side of the greater force.*

From an inspection of the figures of this and the last Article, it is evident that when the two forces act in the same direction, the line of action of their resultant cuts the line joining the points of application of the forces internally; and that when the forces act in opposite directions, the line of action of the resultant cuts the line joining the points of application produced, on the side of the greater force.

30. We may express AC and CB (Arts. 28 and 29) in terms of AB, P, and Q.

In Art. 28 we have

$$P:Q::BC:AC,$$

whence result

$$P:P+Q::BC:(AC+BC=)\,AB,$$

and $$P+Q:Q::(AC+BC=)\,AB:AC;$$

$$\left.\begin{aligned}\therefore\quad &(P+Q)BC=P.AB, \text{ or } BC=\frac{P}{P+Q}.AB\\ \text{and}\quad &(P+Q)AC=Q.AB, \text{ or } AC=\frac{Q}{P+Q}.AB\end{aligned}\right\}\ \ldots\ldots (1).$$

Also in Art. 29 we have

$$P:Q::BC:AC,$$

whence result

$$P:Q-P::BC:(AC-BC=)\,AB,$$

and $$Q-P:Q::(AC-BC=)\,AB:AC;$$

$$\left.\begin{aligned}\therefore\quad &(Q-P)BC=P.AB, \text{ or } BC=\frac{P}{Q-P}.AB\\ \text{and}\quad &(Q-P)AC=Q.AB, \text{ or } AC=\frac{Q}{Q-P}.AB\end{aligned}\right\}\ \ldots\ldots (2).$$

If in (2) we put $Q=P$, we have

$$R=Q-P=0,\quad BC=\infty,\quad AC=\infty.$$

Hence, *two equal parallel forces which act in opposite directions, but not in the same straight line, cannot be replaced by any single finite force acting at a finite distance.*

Such a system of forces is called a *Couple.*

31. Let P_1, P_2, P_3, ... P_n be any number of parallel forces acting in the same direction, and applied at the points A_1, A_2, A_3, ... A_n of a rigid system.

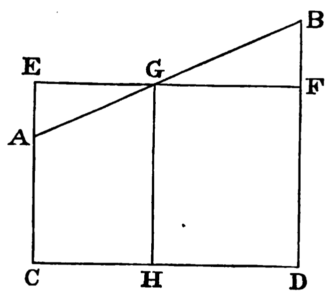
By Art. 28, the point of application G of their resultant will be obtained by compounding P_1 and P_2, then the resultant of P_1 and P_2 with P_3, and so on. The final resultant R of $P_1, P_2, \ldots P_n$ will evidently be parallel to these forces and equal to their sum. Thus the point G depends only on the relative magnitudes of the parallel forces, and on the figure formed by their points of application. Hence we have the following theorem.

If the directions of all the forces $P_1, P_2, \ldots P_n$ *be changed simultaneously in such a manner that they may still pass through the same points of application,* $A_1, A_2, \ldots A_n$, *and preserve their parallelism, the resultant of all these forces will always pass through the same point* G.

The point G is called the *Centre of parallel forces.*

The *moment of a force with respect* to a plane is the product of the force into the distance of its point of application from the plane. If the perpendiculars on one side of the plane be considered positive, those on the other side will be negative.

32. *If two parallel forces* (P_1 *and* P_2) *act in the same direction, the sum of their moments with respect to any plane is equal to the moment of their resultant* (R) *with respect to the same plane.*



Let A, B, and G be the points of application of P_1, P_2, and R.

Draw AC, GH, BD perpendicular to the plane and meeting it in the points C, H, D and draw EGF parallel to CHD.

By Art. 28,

$$R = P_1 + P_2,$$

and $$\frac{AG}{BG} = \frac{P_2}{P_1}.$$

But from the similar triangles AGE, GBF,

$$\frac{AG}{GB}=\frac{AE}{BF}, \quad \therefore \frac{AE}{BF}=\frac{P_2}{P_1}, \text{ and } \therefore P_1 . AE = P_2 . BF;$$

or
$$P_1(GH-AC)=P_2(BD-GH),$$

$$\therefore \quad (P_1+P_2)\,GH = R . GH = P_1 . AC + P_2 . BD.$$

Q. E. D.

33. *If any number of parallel forces* P_1, P_2, ... P_n *act in the same direction, the sum of their moments with respect to any plane is equal to the moment of their resultant* R *with respect to the same plane.*

Let z_1, z_2, ... z_n be the distances of the points of application of the forces from the plane, $\bar{z}$ the distance of the point of application of the resultant from the same plane;

When the forces are compounded, by Art. 28, let R_1, R_2, &c. be the successive resultants, δ_1, δ_2, the distances of the points of application of these resultants from the plane,

then, by Art 32, we have

$$R_1 . \delta_1 = P_1 z_1 + P_2 z_2,$$
$$R_2 . \delta_2 = R_1 . \delta_1 + P_3 z_3 = P_1 z_1 + P_2 z_2 + P_3 z_3,$$

and so on.

Therefore, finally, $R . \bar{z} = P_1 z_1 + P_2 z_2 + \ldots + P_n z_n$,

$$\therefore \quad \bar{z} = \frac{P_1 z_1 + P_2 z_2 + \ldots + P_n z_n}{P_1 + P_2 + \ldots + P_n} = \frac{\Sigma(Pz)}{\Sigma P}, \text{ suppose.}$$

Similarly, if we take the moments with respect to two other planes at right angles to each other and to the first, we shall find

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma P}, \text{ and } \bar{y} = \frac{\Sigma(Py)}{\Sigma P},$$

where x and y denote quantities similar to z. Q. E. D.

$\bar{x}$, $\bar{y}$, $\bar{z}$ fix the position of the centre of parallel forces, which is therefore unique.

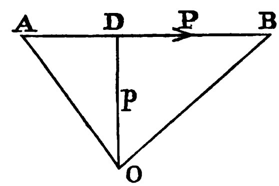
CHAPTER IV.

ON MOMENTS AND CONDITIONS OF EQUILIBRIUM.

34. The product of a force into the perpendiculars on its line of action from any point is called *the moment of the force about the point, or with respect to the point.*

If we suppose the eye placed at the point and looking along the perpendicular from it on the line of action of the force, then the force will tend to twist the body on which it acts, either from left to right or from right to left, about the point, supposed fixed. If one of these directions be considered positive, then the other will be negative. We shall generally consider the direction from left to right as the positive one, that is, these moments will be considered positive, when the forces tend to twist the body in the same direction as the hands of a watch revolve.

35. *To shew that the moment of a force about any point may be represented geometrically by twice the area of the triangle which has the point for its vertex, and the line representing the given force for its base.*



Let AB represent any force P, and let $OD = p$ be drawn perpendicular to AB from the point O; then, by definition,

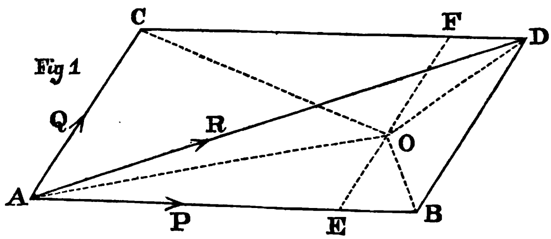
$P.p$ is the moment of P about O;

but $P.p$ or $AB.OD =$ twice the area of the triangle AOB, (Euc. I. 41). Q. E. D.

36. *If two forces act at a point of a rigid body, the algebraical sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.*

Let AB, AC represent the two forces P and Q; then AD the diagonal of the parallelogram $ABDC$ will represent their resultant R.

Fig 1



Let O be the point about which the moments are taken, and through O draw EF parallel to AC.

Q and R tend to twist the body round O from left to right, and P tends to twist it from right to left; therefore the moments of Q and R are positive, and that of P is negative.

The Algebraical sum of the moments of P and Q about O is

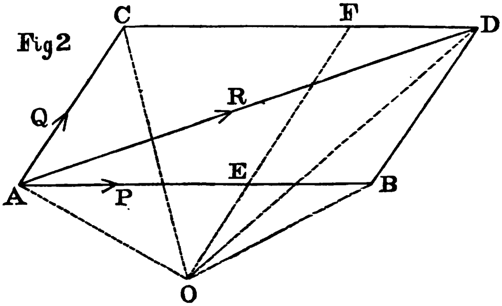
$$=2\triangle AOC-2\triangle AOB=2(\triangle AOC+\triangle BOD)-2\triangle AOB-2\triangle BOD$$

$$=2\triangle ABD-2\triangle AOB-2\triangle BOD \text{ (Euc. I. 41 and 34)}$$

$$=2\triangle AOD$$

$=$ moment of R about O.

This proof is applicable when O lies within the angle BAC or its vertically opposite angle; but if O lie within either of the supplemental angles of BAC, we have (fig. 2), sum of the moments of P and Q about O

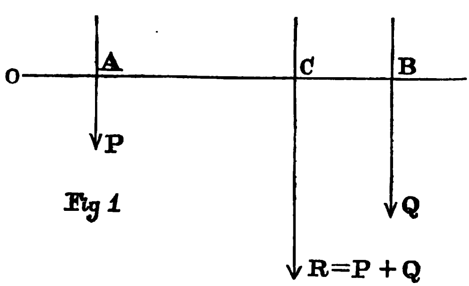


$=2\triangle AOC+2\triangle AOB$
$=2(\triangle AOC+\triangle BOD)+2\triangle AOB-2\triangle BOD$
$=2\triangle ABD+2\triangle AOB-2\triangle BOD$ (Euc. I. 41 and 34)
$=2$ quadrilateral $AOBD-2\triangle BOD$
$=2\triangle AOD$
= the moment of R about O. Q. E. D.

(See Exercises on Euclid, No. 146, p. 135.)

37. *The algebraical sum of the moments of two parallel forces about any point in their plane is equal to the moment of their resultant about the same point.*

Let P and Q be two parallel forces, R their resultant, and O the point about which moments are taken.



Through O draw a perpendicular to the lines of action of the three forces P, Q, R meeting them respectively in the points A, B, C. The three forces may be supposed to act at these points.

Thus, in fig. 1, we have

the sum of the moments of P and Q about O

$= Q.OB + P.OA = Q.(OC + CB) + P.(OC - AC)$

$= (P + Q).OC + Q.CB - P.AC$

$= R.OC$, since $Q.CB - P.AC = O$, by Art. 28,

$=$ the moment of R about O.

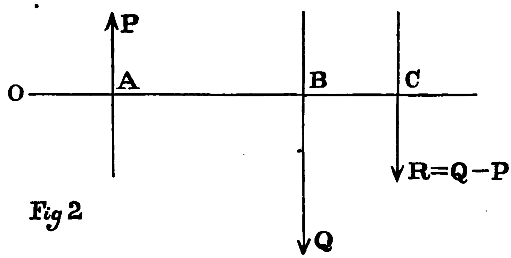


Fig 2

Again, in fig. 2, we have

the sum of the moments of P and Q about O

$= Q.OB - P.OA = Q.(OC - CB) - P(OC - AC)$

$= (Q - P).OC - (Q.CB - P.AC)$.

$= R.OC$, since $Q.CB - P.AC = O$, by Art. 29,

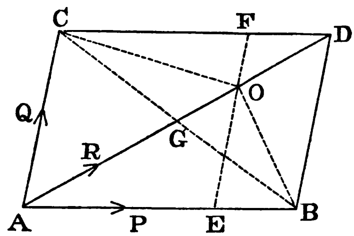
$=$ the moment of R about O. Q. E. D.

If $Q = P$, then the two forces constitute a couple, and the sum of the moments of the two forces of the couple about any point O in its plane $= P.OB - P.OA = P(OB - OA) = P.AB$, which is constant, and called *the moment of the couple.* Hence, *the sum of the moments of the forces of a couple about any point in the plane of the couple is constant.*

The perpendicular distance between the lines of action of the two forces of a couple is called the *Arm of the couple.*

38. In Arts. 36 and 37, if the point O be taken anywhere on the line of action of the resultant R, then the moment of R is zero, and therefore *the moments of any two forces about any point on the line of action of their resultant are equal in magnitude but opposite in sign.*

In the case of parallel forces this also follows directly from Arts. 28 and 29. For forces acting at a point it may be proved directly as follows.



Draw FOE parallel to AC; then

$$\triangle AOC + \triangle BOD = \triangle ABD \text{ (Euc. I. 41 and 34)}$$
$$= \triangle AOB + \triangle BOD;$$

$$\therefore\ 2\triangle AOC = 2\triangle AOB,$$

that is, the moment of Q about O = the moment of P about O.

Q. E. D.

Otherwise thus [by Art. 13 (3)] $CG, = GB$; therefore

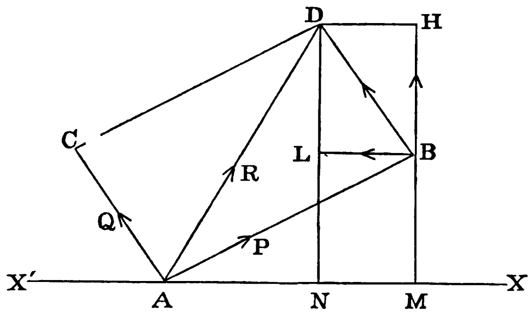
$$\triangle AGC = \triangle AGB \text{ and } \triangle CGO = \triangle OGB \text{ (Euc. I. 38)},$$

therefore $2\triangle AOC = 2\triangle AOB$,

that is, the moments of P and Q about O are equal and of contrary signs. Q. E. D.

39. *To show that, if two forces act in the same plane, the algebraical sum of their resolved parts along any straight line in their plane is equal to the resolved part of their resultant along the same straight line.*

Let AB, AC represent two forces P and Q, then AD the diagonal of the parallelogram $ABDC$ will represent their resultant.



Let $X'AX$ be the assigned direction.

Draw BM and DN perpendicular to $X'X$, and BL parallel to it. Complete the rectangle LH.

BL and BH are clearly the resolved parts of a force represented by BD, in the direction XX' and perpendicular to XX', but AC and BD are equal and parallel; therefore BL or MN represents the resolved part of AC along $X'X$ in direction and magnitude. It is also clear that AM is the resolved part of P and AN of R along $X'X$.

Now the forces AM and MN act in opposite directions, and therefore their algebraical sum is AN, which is the resolved part of AD along $X'X$.

Thus the proposition is true when the two forces intersect.

Again, when the forces are parallel, it is obvious that their resolved parts and that of their resultant, will be the same as if they all acted in the same straight line, retaining their original directions and magnitudes. This case will be now evident on making a figure.

Hence, *the sum of the resolved parts of the two forces of a couple in any direction in their plane is manifestly zero.*

Q. E. D.

40. *A couple and a single force acting in the same plane on a rigid body can be reduced to a single resultant force.*

For if the three forces be parallel, the single force must act in the same direction as one of the forces of the couple, and therefore these two forces can be compounded into a force greater than the remaining force of the couple, and parallel to it, but acting in the opposite direction. This resultant and the remaining force of the couple can now be compounded into a single force, which is the final resultant of the three parallel forces; but if the single force be not parallel to the forces of the couple, it can be compounded with either force of the couple into a single force, and this force can be again compounded with the remaining force of the couple into a single force, which is the final resultant. Q. E. D.

41. *A system of forces acting on a rigid body in the same plane can be reduced to a single resultant force, or to a single resultant couple.*

For if the first two forces constitute a couple, they can be combined with the third force into a single resultant force; but if the first two forces do not form a couple, they can be replaced by their resultant: either of the resultants just found can be combined with the remaining forces of the system, and in this way we clearly arrive at a single resultant force, or a resultant couple. Q. E. D.

42. *If a system of forces act upon a rigid body in one plane, the algebraical sum of the resolved parts of the forces in any direction in their plane is equal to the resolved part of the resultant in the same direction.*

Let the forces be P_1, P_2, P_3 ... P_n, and suppose that they have been compounded in this order according to the method adopted in the last Article.

Let R_1 be the resultant of P_1 and P_2,
R_2 . . . . of R_1 and P_3,
and so on. Also let Q be the resultant of the first $n-1$ forces.

By Art. 39, the sum of the resolved parts of P_1 and P_2 in any given direction is equal to the resolved part of R_1 in the same direction. Again, the sum of the resolved parts of R_1 and P_3, *i.e.* of P_1, P_2, and P_3 in the given direction, is equal to the resolved part of R_2 in the same direction, and so on.

Thus the sum of the resolved parts of $P_1, P_2 \dots P_{n-1}$ in the given direction is equal to the resolved part of Q in the same direction. Now, if Q and P_n have a single resultant R, the sum of the resolved parts of Q and P_n in the given direction is equal to the resolved part of R, that is, the sum of the resolved parts of all the forces of the system in the given direction is equal to the resolved part of their single resultant in the same direction.

But if Q and P_n constitute the final resultant couple, then, since the sum of the resolved parts of the forces $P_1, P_2 \dots P_{n-1}$ in the given direction is equal to the resolved part of Q in the same direction, adding to these equals the resolved part of P_n in the given direction, we have
the sum of the resolved parts of all the forces of the system equal to the sum of the resolved parts of the forces of the resultant couple; that is, equal to zero. Q. E. D.

43. *If a system of forces act upon a rigid body in one plane, the algebraical sum of their moments about any point in the plane is equal to the moment of their resultant about the same point.*

This is proved in precisely the same manner as the theorem in the last Article, by the help of Arts. 36 and 37.

44. *To find the conditions of equilibrium of a system of forces acting on a rigid body in one plane.*

Since (Art. 42) the sums of the resolved parts of the system of forces in any two directions in their plane at right angles to one another, are respectively equal to the resolved parts of the resultant in the same direction; therefore for equilibrium (as then the resultant is zero) these sums must be separately zero; but these conditions though *necessary*, are not *sufficient*, since we know (Art. 39) that they are satisfied when the system of forces reduces to a resultant couple.

Now (Art. 43) as the algebraical sums of the moments of the forces is always equal to the moment of the resultant, and as the moment of a couple is constant, we must also have for equilibrium the condition that the algebraical sum of the moments of the forces about any point in their plane must be zero.

Hence, *the three necessary and sufficient conditions, in order that any system of forces acting upon a rigid body in one plane should be in equilibrium, are, that the algebraical sums of the resolved parts of the forces in any two directions in their plane at right angles to one another should be separately zero, and that the algebraical sum of their moments about any point in their plane should also be zero.*

45. The conditions of equilibrium obtained in the last Article may be exhibited in a different form.

If the sums of the moments of the forces about any three points, A, B, C, in the plane of the forces and not lying on the same straight line, be separately zero, the resultant must be zero, and therefore the system of forces must be in equilibrium.

For if the resultant be not zero, the point A lies on the line of action of the resultant, since the sum of the moments of the forces about any point in their plane is equal to the moment of the resultant about the same point.

Similarly, the points B and C lie on the line of action of the resultant; therefore the resultant acts along all the three sides of the triangle ABC, which is absurd.

Hence, the resultant must be zero, and the system of forces must therefore be in equilibrium.

Hence, the three conditions of equilibrium of a system of forces acting in one plane on a rigid body are these—*The algebraical sum of the moments of the forces about any three points in their plane, and not lying in the same straight line, must be separately zero.*

46. *Three Forces in Equilibrium.*

If three forces maintain a rigid body in equilibrium, it is clear that one of them must be equal and opposite to the resultant of the other two, and act in the same straight line. Hence, if the lines of action of two of the forces intersect, the third force must pass through the point of intersection; and if two of the forces be parallel, the third force must be parallel to them.

Also, the moments of any two of the forces about any point in the line of action of the third must be equal and of contrary signs; and in the case in which the lines of action of the three forces meet in a point, each force must be proportional to the sine of the angle between the other two; and if the forces be represented by straight lines drawn from the common point on their lines of action, each force must be *without* the angle ($< 180°$) contained by the other two.

CHAPTER V.

ON THE CENTRE OF GRAVITY.

47. The force with which all bodies are attracted towards the surface of the earth is called *gravity*.

It is exerted upon every material particle in a direction perpendicular to the surface of still water. This direction is called the *vertical* direction, and a plane perpendicular to the vertical is called a *horizontal* plane. The vertical at any place is also the direction in which a body falls freely, or the direction of the plumb-line.

As the bodies which we shall consider have always very small dimensions, compared with the radius of the earth, we may regard the verticals drawn from the different points of the same body as parallel to each other.

The intensity of gravity varies with the latitude and with the height of the body, but we may, without appreciable error, suppose this intensity constant at different points of the same body.

48. A body may therefore be considered as an assemblage of an indefinite number of particles, which are invariably connected together and acted on by small parallel forces in the same vertical direction.

By Arts. 31 and 33, these parallel forces, in every position of the body, will have a resultant equal to their sum, and this resultant will always act vertically through the *same* point. This point is called the *centre of gravity* of the body, and the resultant is called the *weight* of the body.

It may be useful to give formal definitions of these terms.

The *weight* of a body is the resultant of the Earth's attraction on all the particles of the body.

The point through which the resultant of the Earth's attraction on any body always acts, in every position of the body, is called the *centre of gravity* of the body.

49. Since the weight of a body, in every position of the body, will always act through its centre of gravity, the body will remain in equilibrium, if the centre of gravity be fixed. The whole weight of the body may therefore be supposed collected at the centre of gravity.

50. A body is said to be of *uniform density* when the weight of any portion of it, however small, is to the weight of the whole body as the volume of that portion is to the volume of the whole body. A body of uniform density is also often called a *homogeneous* body. The density of such a body is measured by the ratio of the weight of any volume of it to the weight of an equal volume of some standard substance of uniform density.

[The subject of density will be more fully treated in the HYDROSTATICS.]

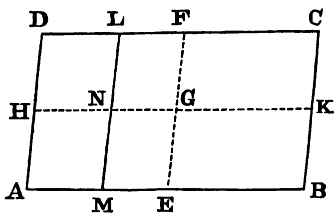
A *lamina* is a plate, or a thin piece of metal, wood, or other material.

51. *To find the centre of gravity of a material straight line (or rod) of uniform density and thickness.*

Since the line is symmetrical on each side of its middle point, the middle point must be its centre of gravity (or the point about which it will balance in all positions), since there is no reason why the centre of gravity should be on one side of the middle point rather than on the other.

Otherwise, thus. Since the rod is of uniform density and thickness, we may suppose it composed of pairs of particles of equal weight, equidistant from the middle point. The middle point will therefore (Art. 28) be the centre of gravity of each pair, and consequently of all the particles, that is, of the line itself. Q. E. F.

52. *To find the centre of gravity of a parallelogram consisting of a lamina of uniform density and thickness.*



Let $ABCD$ be the parallelogram.

Draw EF, HK joining the middle points of its opposite sides, and intersecting in G. G will be the centre of gravity required.

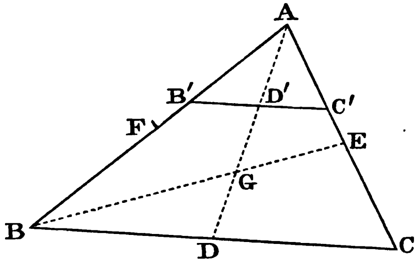
For the parallelogram may be considered as made up of an indefinite number of uniform lines, such as LM, parallel to AD. All these lines will be bisected by HK, and therefore (Art. 51) their centres of gravity will lie in HK. Therefore the centre of gravity of the whole figure lies in HK. Similarly, the centre of gravity lies in EF. Therefore G is the required centre of gravity. Q. E. F.

It is very easy to prove that G is also the intersection of the diagonals of the parallelogram.

53. *To find the centre of gravity of a triangular lamina of uniform density and thickness.*

Let ABC be the triangle, D, E, F, the middle points of its sides.

The triangle may be considered as composed of an in-



definite number of uniform lines, such as $B'C'$, parallel to BC. All these lines will (by Lemma below) be bisected by AD, and therefore (Art. 51) their centres of gravity will lie in AD. Therefore the centre of gravity of the triangle lies in AD. Similarly, it lies in BE, and therefore it is at G, the point of intersection of AD and BE.

Now [Art. 13 (4)] $DG = \frac{1}{3}AD$.

Hence, *the centre of gravity of a triangle lies on the bisector of any side at a distance from the middle point of the side equal to one third of the bisector.*

LEMMA. *Every parallel to a side of a triangle meeting the other two sides is bisected by the bisector of the third side.*

Because $B'C'$ is parallel to BC, the triangles $AD'B'$, ADB are equiangular, and therefore (Euc. VI. 4)

$$AD' : D'B' :: AD : DB;$$

or alternately, $AD' : AD :: D'B' : DB$.

Similarly, $AD' : AD :: D'C' : DC$.

Therefore $D'B' : DB :: D'C' : DC$,

or alternately, $D'B' : D'C' :: DB : DC$;

but $DB = DC$, therefore $D'B' = D'C'$, or $B'C'$ is bisected by AD. Q. E. D.

[In writing out the above proposition at an examination, both this Lemma and the proposition Art. 13 (4), namely, that $DG = \frac{1}{3}AD$, should be also written out or included.]

54. *To shew that the centre of gravity of three equal particles placed at the vertices of a triangle coincides with the centre of gravity of the triangle.*

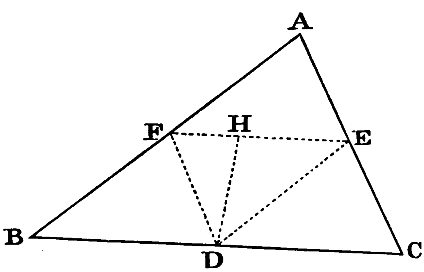
(Fig. Art. 53.)

Let P be the weight of each of the three particles which are placed at A, B, C respectively, and G the centre of gravity of the triangle ABC.

Because $BD = DC$, D is the centre of gravity of the two equal particles at B and C, and these particles may therefore be replaced by $2P$ at D; but since G is the centre of gravity of the triangle ABC, $DG : GA :: 1 : 2$, that is, DA is divided inversely as the two weights at D and A; therefore (Art. 28) G is the centre of gravity of these two weights, and therefore of the three equal weights at A, B, C. Q. E. D.

55. *A triangle is formed by straight lines of uniform density and thickness. Shew that the centre of gravity of the perimeter coincides with the centre of the circle inscribed in the triangle formed by joining the middle points of the sides of the given triangle.*

Let ABC be the given triangle, D, E, F, the middle



points of its sides.

The weights of the three sides are proportional to their lengths, and may be supposed to act at their middle points D, E, F, (Art. 51).

Divide FE in H, so that

$$FH : HE :: AC : AB \qquad \text{.................. (1)};$$

then (Art. 28) H is the centre of gravity of AB and AC.

Therefore the centre of gravity of the perimeter will lie in DH.

Again [Art. 13 (4)], $DF = \frac{1}{2}AC$, and $DE = \frac{1}{2}AB$, therefore (1) becomes

$$FH : HE :: 2DF : 2DE$$
$$= DF : DE;$$

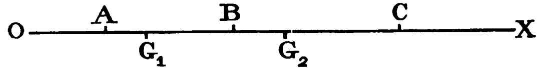
therefore (Euc. VI. 3) DH bisects the angle EDF.

Similarly, the bisectors of the other two angles of the triangle DEF pass through the centre of gravity of the perimeter of the triangle ABC; but the bisectors of the angles of a triangle meet in the centre of its inscribed circle (Euc. IV. 4).

Hence, the centre of the inscribed circle of the triangle DEF, is the centre of gravity of the perimeter of the triangle ABC. Q. E. D.

56. *To find the centre of gravity of several particles lying in a straight line.*

Let A, B, C, &c. be the positions of the particles lying in



the straight line OX,

$P_1, P_2, P_3, \ldots P_n$ their weights, and

$x_1, x_2, x_3, \ldots x_n$ their distances from the fixed point O.

Divide AB in G_1, so that

$$P_2 : P_1 :: AG_1 : BG_1 \qquad \text{.................. (1)};$$

then G_1 is the centre of gravity of P_1 and P_2, and $P_1 + P_2$ may therefore be considered as collected at G_1.

From (1) we have

$$P_1 . AG_1 = P_2 . BG_1,$$

or $$P_1(OG_1 - x_1) = P_2(x_2 - OG_1);$$

$\therefore$ $$(P_1 + P_2) OG_1 = P_1x_1 + P_2x_2 \text{ .... (2).}$$

Similarly, if G_2 be the centre of gravity of $P_1 + P_2$ at G_1 and P_3 at C, we shall find

$$(\overline{P_1 + P_2} + P_3) OG_2 = (P_1 + P_2) OG_1 + P_3x_3,$$

or by (2) $$(P_1 + P_2 + P_3) OG_2 = P_1x_1 + P_2x_2 + P_3x_3,$$

and so on.

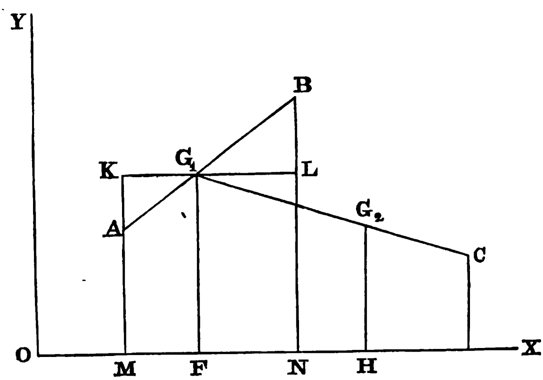
Hence, if G be the centre of gravity of all the particles, and if we put $\bar{x}$ for OG, we shall have

$$(P_1 + P_2 + P_3 + \ldots + P_n)\bar{x} = P_1x_1 + P_2x_2 + \&c. + P_nx_n;$$

$$\therefore \quad \bar{x} = \frac{P_1x_1 + P_2x_2 + \ldots + P_nx_n}{P_1 + P_2 \ldots + P_n} = \frac{\Sigma(Px)}{\Sigma(P)}, \text{ suppose.} \qquad \text{Q. E. F.}$$

57. *To find the centre of gravity of several particles lying in a plane.*

Let $A, B, C \ldots,$ be the positions of the particles lying in



the plane of the paper, $P_1, P_2, P_3 \ldots, P_n$ their weights, and OX, OY two fixed straight lines intersecting at right angles in O.

Let the distances of A, B, C ..., from OX be $y_1, y_2, \dots y_n$, and let the distances from OY be $x_1, x_2, \dots x_n$.

Divide AB in G_1, so that

$$P_2 : P_1 :: AG_1 : BG_1 \quad \text{.......................(1)},$$

then G_1 is the centre of gravity of P_1 and P_2, and $P_1 + P_2$ may therefore be considered as collected at G_1.

Draw AM, G_1F, BN perpendicular to OX, and KG_1L parallel to OX; then from the similar triangles AG_1K, BG_1L we have $AG_1 : BG_1 :: AK : BL$;

therefore, by (1), $P_2 : P_1 :: AK : BL$;

therefore $\quad P_1 . AK = P_2 . BL,$

or $\quad P_1(G_1F - AM) = P_2(BN - G_1F),$

that is, $\quad P_1(G_1F - y_1) = P_2(y_2 - G_1F)$;

therefore $\quad (P_1 + P_2)\, G_1F = P_1y_1 + P_2y_2$ ....(2).

Again, let G_2 be the centre of gravity of $P_1 + P_2$ at G_1 and P_3 at C, and draw G_2H perpendicular to OX;

then by (2) we have

$$(\overline{P_1 + P_2} + P_3)\, G_2H = (P_1 + P_2)\, G_1F + P_3y_3,$$

or

$$(P_1 + P_2 + P_3)\, G_2H = P_1y_1 + P_2y_2 + P_3y_3,$$

and so on.

Hence, if G be the centre of gravity of all the particles, and $\bar{y}$ the distance of G from OX, we have

$$(P_1 + P_2 + \dots + P_n)\bar{y} = P_1y_1 + P_2y_2 + \dots + P_ny_n,$$

$$\therefore \quad \bar{y} = \frac{P_1y_1 + P_2y_2 + \dots + P_ny_n}{P_1 + P_2 + \dots + P_n} = \frac{\Sigma(Py)}{\Sigma(P)}.$$

Similarly, $$\bar{x} = \frac{\Sigma(Px)}{\Sigma(P)},$$

where $\bar{x}$ is the distance of G from OY.

$\bar{x}$ and $\bar{y}$ completely determine the position of the centre of gravity. Q. E. F.

58. *To find the centre of gravity of several particles arranged in any manner in space.*

This problem has been completely solved in Art. 33, and it includes, as particular cases, the problems of the last two Articles. It also enables us *to find the centre of gravity of a system of bodies arranged in any manner in space*, when we know the weight and the position of the centre of gravity of each body.

59. *Given the positions of the centres of gravity of a body and of a portion of it; to find the position of the centre of gravity of the other portion.*

Let A and B be the given centres of gravity of the whole body and of a portion of it, W and W_1 their weights; then $W - W_1$ is the weight of the other portion.

Produce BA to C, so that

$$BA : AC :: W - W_1 : W_1;$$

therefore $$AC = \frac{W_1}{W - W_1} \cdot AB,$$

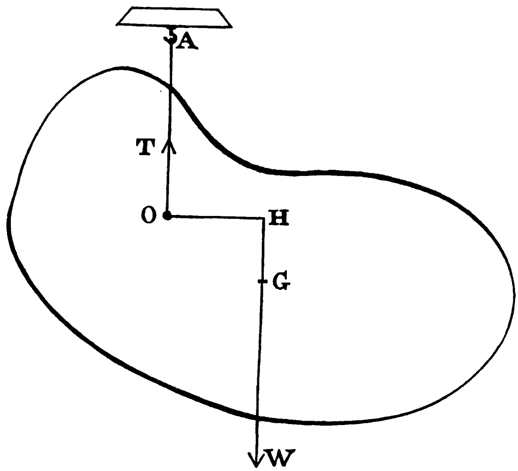
which determines the required position of C, the centre of gravity of the other portion. Q. E. F.

60. *When a body is suspended from a point about which it can turn freely, it will rest in equilibrium with its centre of gravity in the vertical line passing through the point of suspension.*

Let O be the point of suspension, G the centre of gravity of the body, and W its weight.

When the body is in equilibrium, let G, if possible, not be in the vertical through O. Draw OH perpendicular to the vertical through G; then the sum of the moments about

O of the forces acting upon the body is $W.OH$, which does



not vanish, and therefore (Art. 44) the body cannot be in equilibrium, which is contrary to the hypothesis.

Therefore G must be in the vertical through O. Q. E. D.

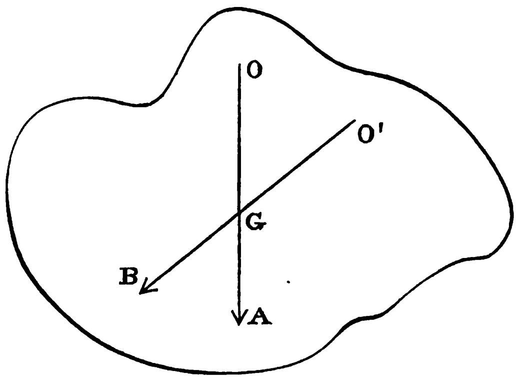
The pressure on the point O is clearly equal to W.

Otherwise thus. *When a body can turn freely about a fixed point, it will rest in equilibrium with its centre of gravity either vertically* above *or vertically* below *the fixed point.*

For the only two forces acting upon the body are the reaction of the fixed point and the weight acting in the vertical through the centre of gravity. It is clear that these two forces cannot equilibrate one another, unless they are equal and opposite and act in the same straight line. Hence, the centre of gravity must lie in the vertical through the fixed point. Q. E. D.

61. *To determine, by experiment, the centre of gravity of a lamina of any form.*

Suspend the lamina by a cord from any point O in its surface, then its centre of gravity will lie in the vertical line



through O, when it rests in equilibrium (Art. 60).

Let OA be this vertical line through O, traced on the surface of the lamina.

Again, suspend the lamina from another point O' not lying in the straight line OA, and let $O'B$ be the vertical line traced on the surface of the lamina, when it rests in equilibrium in its second position.

Since OA and $O'B$ each pass through the centre of gravity of the lamina, their point of intersection G must be the required centre of gravity. Q. E. F.

The same method will, of course, apply to a body of any form if we can trace the vertical lines through the points of suspension.

62. When a body in equilibrium under the action of any forces receives a slight displacement, if the forces tend to bring it back to its original position of equilibrium, this position was one of *stable* equilibrium; but if the forces tend

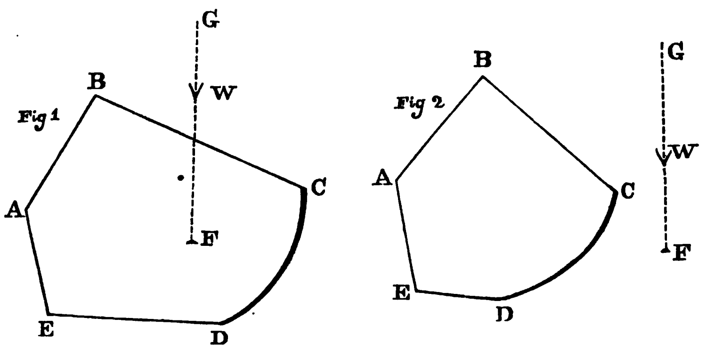
to remove it still further from its original position, this position was one of *unstable* equilibrium.

When the body remains in equilibrium in its new position, the original position was one of *neutral* equilibrium.

63. *When a body is placed on a horizontal plane it will stand or fall over, according as the vertical through its centre of gravity falls within or without the base.*

[By the base is meant the figure formed by drawing a fine string tightly round the exterior points of the body, which are in contact with the plane, so that some of the points of contact of the body with the plane may be *within* this figure; the base is supposed to be in the plane of the paper, and G above this plane.]

Let $ABCDE$ be the base of the body, G its centre of



gravity, and let the vertical through G meet the plane in F.

The reaction of the plane on the body is called into play solely by the downward pressure of the body on the plane; hence, since action and reaction are equal and opposite, the reaction of the plane on the body will be equal and opposite to the weight of the body. The resultant reaction of the plane will therefore act vertically upwards through some point within the base, and be equal to W. Call this reaction R.

Since W and R are the only two forces acting on the body, there can only be equilibrium when W and R act in the same straight line, which is impossible when F is outside the base, as in fig. 2.

In fig. 1, W and R will act in the same vertical line, and will therefore be in equilibrium; for if we suppose the body turned through a very small angle about any point (or straight line) of its base, as A, then the moment of W about A will tend to bring the body back to its original position, which was therefore one of stable equilibrium, and therefore W and R destroy one another. Q. E. D.

[In the case of fig. 1, it is easy to shew that when one or two points of the body are in contact with the plane, the pressures on the points are determinate: when three points are in contact the equilibrium is stable, and the pressure on each point determinate, but when more than three points are in contact, though the equilibrium is stable, the pressure on each point is indeterminate. We might also investigate the point or line about which the body in fig. 2 would *begin* to turn, but such investigations are beyond the scope of the present work.]

64. *To find the centre of gravity of a homogeneous pyramid on a triangular base.*

Let ABC be the base of the pyramid, D its vertex.

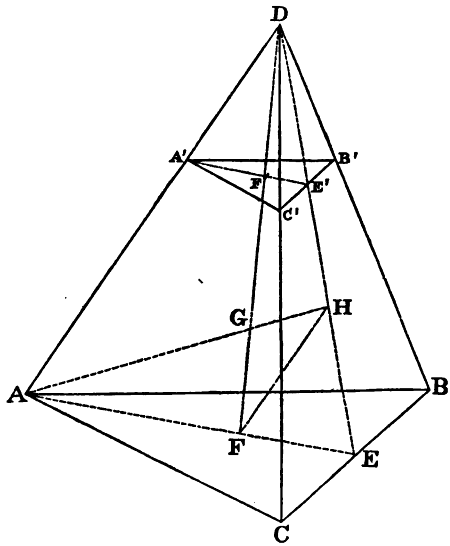
Bisect BC in E, and join AE, DE.

Take $EF = \frac{1}{3}AE$, and $EH = \frac{1}{3}DE$; then F and H are the centres of gravity of the triangles ABC, CDB.

Join AH, FD intersecting in G.

G will be the required centre of gravity.

For the pyramid, being homogeneous, may be considered



as composed of an indefinite number of uniform laminæ parallel to the base ABC.

Let $A'B'C'$ be one of these laminæ.

Because the two parallel planes ABC, $A'B'C'$ are cut by the plane CDB in the straight lines BC, $B'C'$, therefore BC, $B'C'$ are parallel (Euc. XI. 16).

Similarly, $A'B'$, $A'E'$, $A'C'$, are respectively parallel to AB, AE, AC.

Because BC and $B'C'$ are parallel, and $CE = EB$, therefore (Art. 53, Lemma) $C'E' = E'B'$.

Also, because the triangles AFD and $A'F'D$, EFD and $E'F'D$ are equiangular, therefore $AF : FD :: A'F' : F'D$,

or alternately, $AF : A'F' :: FD : F'D$,
and likewise, $EF : E'F' :: FD : F'D$,
therefore, $AF : A'F' :: EF : E'F'$,
or alternately, $AF : EF :: A'F' : E'F'$;
but $EF = \frac{1}{2}AF$, therefore $E'F' = \frac{1}{2}A'F' = \frac{1}{3}A'E'$;
therefore F' is the centre of gravity of the triangle $A'B'C'$.

In the same manner it can be shewn that every other lamina parallel to ABC has its centre of gravity in DF. Therefore the centre of gravity of the whole pyramid is in DF. Similarly, the centre of gravity of the pyramid is also in AH; therefore it is at G the point of intersection of AH and DF.

Join FH. Because $EF : FA :: EH : HD = 1 : 2$, therefore (Euc. VI. 2) FH and AD are parallel, and therefore

$$EF : FH :: EA : AD,$$

or alternately, $EF : EA :: FH : AD$;
but $EF = \frac{1}{3}EA$, therefore $FH = \frac{1}{3}AD$.

Again, from the equiangular triangles FGH, DGA,

$$HF : FG :: AD : DG,$$

or alternately, $HF : AD :: FG : DG$;
but $HF = \frac{1}{3}AD$, therefore $FG = \frac{1}{3}DG = \frac{1}{4}DF$.

Hence, *the centre of gravity of a triangular pyramid lies on the straight line joining a vertex with the centre of gravity of the opposite face, at a distance from the vertex equal to three fourths of the joining line.*

65. The learner may prove the following results:—

(1) *The centre of gravity of a pyramid on any plane base lies in the straight line joining its vertex to the centre of gravity of its base, at a distance from the vertex equal to three fourths of the joining line.*

(2) *The centre of gravity of a right cone lies on its axis at a distance from the vertex equal to three fourths of the axis.*

(3) *The centre of gravity of the curved surface of a right cone lies on its axis at a distance from the vertex equal to two thirds of the axis.*

(4) *The centre of gravity of four equal particles placed at the vertices of a triangular pyramid coincides with the centre of gravity of the pyramid.* (See Art. 54.)

The pyramid and cone are supposed to be homogeneous, and the surface of the cone is supposed to be a uniform lamina.

CHAPTER VI.

ON THE MECHANICAL POWERS.

66. A *Machine* is an instrument for transmitting a force applied at one point to some other point, or for changing the direction or intensity of a force.

The simplest parts of a machine consist of cords, rods, fixed points, or fixed surfaces.

Fixed points can resist pressure in all directions, *smooth* fixed lines and fixed surfaces only in directions perpendicular (or what is the same thing, *normal*) to them.

The force (P) applied to a machine in equilibrium is generally called the *Power*, and the resistance (W) to be overcome by P is called the *Weight*.

There are six simple Machines, which are generally called the *Mechanical Powers*, viz.—the *Lever*, the *Wheel and Axle*, the *Pulley*, the *Inclined Plane*, the *Screw*, and the *Wedge*.

By a combination of the Mechanical Powers, all machines, however complex, are formed.

67. THE LEVER. A *Lever* is a rigid rod or bar capable of motion about a fixed point or a fixed axis, which is called the *fulcrum*.

P and W are applied at different points of the lever, and their distances (measured along the lever) from the fulcrum, are called the *Arms* of the lever.

When the arms of a lever are in the same straight line, it is called a *Straight* lever; when the arms are not in the same straight line, it is called a *Bent* lever.

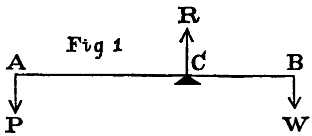
A Straight lever, supposed to be without weight, is sometimes called a *Simple* lever.

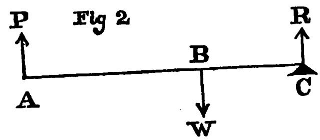
68. Straight Levers are divided into *three kinds*.

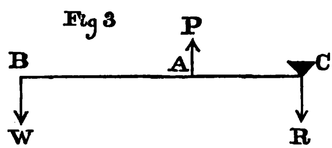
In the *first* kind of lever the fulcrum (C) is between P and W, as in fig. 1.

In the *second* kind of lever W is between P and the fulcrum, as in fig. 2.

In the *third* kind of lever P is between W and the fulcrum, as in fig. 3.







69. *To find the condition of equilibrium in a simple lever and the pressure on the fulcrum, when* P *and* W *are parallel.* (See figs. Art. 68).

Let R be the pressure on the fulcrum C.

It is clear that when there is equilibrium, the resultant of P and W must pass through the fulcrum C and be destroyed by its reaction, therefore (Arts. 28, 29)

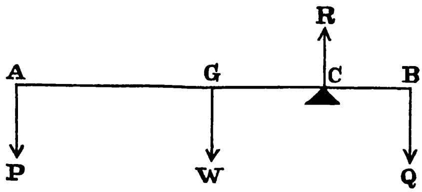
$$P : W :: BC : AC,$$

that is, $P : W$ *inversely as the arms at which they act,*

and
$$R = P + W \text{ in fig. } 1,$$
$$R = W - P \text{ in fig. } 2,$$
$$R = P - W \text{ in fig. } 3.$$

70. *If two weights balance one another on a straight lever in any position which is not vertical, they will balance in any other position of the lever.*

Let AB be the lever in its first position, which, by



hypothesis, is one of equilibrium, G its centre of gravity, W its weight, and P and Q the two weights suspended from the points A and B.

Since the lever is in equilibrium under the action of the three parallel forces P, Q, W, and the reaction R of the fulcrum C, therefore (Art. 31) the resultant of P, Q, W must pass through C, and we must have

$$R = P + Q + W \quad \text{..................} \quad (1)$$

If the lever be turned about C into any other position, then P, Q, W will still act in the same vertical direction, and therefore (Art. 31) their resultant will pass through the same point C, and be destroyed by the reaction of this fixed point. Hence equilibrium will still subsist, and the pressure on the fulcrum will be the same as before. Q. E. D.

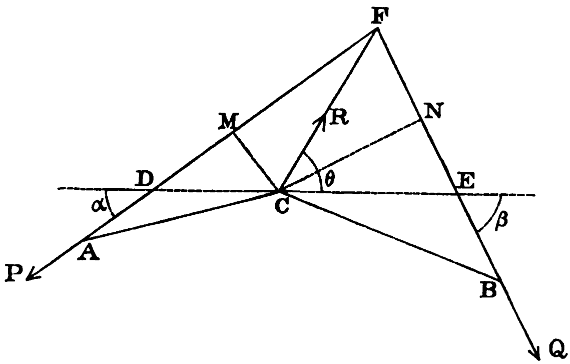
If we take moments about C in any position of the lever, not vertical, since the perpendiculars from C on the lines of action of P, Q, W are clearly proportional to CA, CB, CG, we have $Q.CB - P.CA - W.CG = 0$,

or $$Q.CB = P.CA + W.CG \quad \text{................} \quad (2).$$

(2) gives the condition of equilibrium of a *heavy* straight lever acted on by any two parallel forces.

71. *To find the condition of equilibrium of a lever of any form, without weight, acted on by two forces in any directions in the same plane.*

Let ACB be the lever, C the fulcrum, P and Q the two



given forces acting at A and B, and R the pressure on C.

Produce the lines of action of P and Q to meet in F, then FC must be the line of action of R, when the lever is in equilibrium.

Draw CM, CN perpendicular to AF, BF.

Since there is equilibrium, the moments of P and Q about C must be equal and of contrary signs;

therefore $$P \cdot CM = Q \cdot CN,$$

or $$\frac{P}{Q} = \frac{CN}{CM} \quad \text{..........................(1).}$$

Hence, the required condition of equilibrium is that *the two forces should be inversely as the perpendiculars on their lines of action from the fulcrum, and that they should tend to turn the lever in opposite directions about the fulcrum.*

Q. E. F.

*Let the directions of P and Q make angles α, β with any fixed line DE drawn through C, and let the angle $FCE = \theta$,

then (Art. 22) $R^2 = P^2 + Q^2 + 2PQ\cos(P, Q)$

$$= P^2 + Q^2 - 2PQ\cos(\alpha + \beta),$$

which gives the pressure on the fulcrum.

[The pressure of the fulcrum on the lever is represented in the figure.]

Also (see *Miscellaneous Examples*, 1, N.B.) we have, by Trigonometry,

$$\cot\theta = \frac{CE\cot\alpha - CD\cot\beta}{DE} \text{ .... } (2).$$

or we may find θ thus:

resolving along and perpendicular to DE we have, for equilibrium,

$$P\cos\alpha - Q\cos\beta - R\cos\theta = 0, \text{ or } R\cos\theta = P\cos\alpha - Q\cos\beta \text{......}(3),$$

$$P\sin\alpha + Q\sin\beta - R\sin\theta = 0, \text{ or } R\sin\theta = P\sin\alpha + Q\sin\beta \text{......}(4),$$

$$\therefore \frac{R\sin\theta}{R\cos\theta}, \text{ or } \tan\theta = \frac{P\sin\alpha + Q\sin\beta}{P\cos\alpha - Q\cos\beta} \text{....................}(5).$$

It is easy to show that (2) and (5) are equivalent; for, by (1), we may substitute in (5),

$$CN = CE\sin\beta \text{ for } P \text{ and } CM = CD\sin\alpha \text{ for } Q;$$

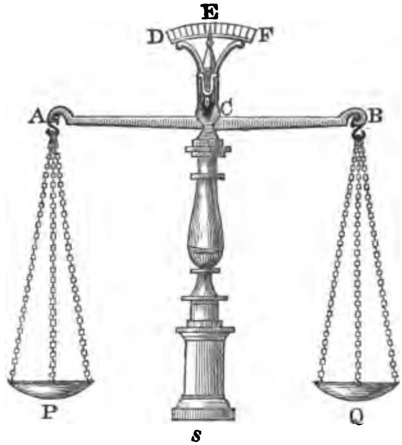
thus (5) becomes $\tan\theta = \dfrac{CE\sin\beta\sin\alpha + CD\sin\alpha\sin\beta}{CE\cos\alpha\sin\beta - CD\sin\alpha\cos\beta}$

$$= \frac{DE}{CE\cot\alpha - CD\cot\beta};$$

therefore $\cot\theta = \dfrac{CE\cot\alpha - CD\cot\beta}{DE}$,

which is identical with (2).

72. The Common Balance.



The annexed figure represents one variety of the *Common Balance*, which is an instrument for weighing bodies. It is essentially composed of a lever (*AB*) called a *beam*, which has a fulcrum (*C*) at its middle point, and two scale-pans (*P* and *Q*) suspended from its extremities.

The fulcrum should be at the middle point of the beam, and the beam should turn very easily about the fulcrum, as it oscillates on each side of its position of equilibrium. For this purpose it is furnished with a *knife-edge* of steel fixed transversely at its middle (*C*) and projecting on both sides. This knife-edge is turned downwards, and rests upon two small plates of steel or agate, which are placed horizontally, the one before, the other behind the beam, and fixed to a solid pedestal or pillar (*S*).

The oscillations of the beam are performed about this knife-edge, as an axis of rotation. The two extremities of the beam are also furnished with two knife-edges, turned upwards. These knife-edges support the hooks to which the chains supporting the scale-pans are attached. When the balance is *just*, the weights placed in the two scale-pans are always equal, if the beam assume a horizontal position.

The requisites of a just balance are these two:

(1) *The distances of the fulcrum from the points of suspension of the scale-pans should be equal.*

(2) *When no body is placed in the scale-pans the beam should be horizontal.*

When these conditions are satisfied and the beam remains horizontal after we have put two bodies in the scale-pans, the weights of these bodies must be equal, since these weights are two forces which equilibrate one another by acting upon the beam at the extremities of equal arms. We may ascertain whether a balance is just in the following manner:

(1) *Ascertain that the beam maintains a horizontal position when the scale-pans contain no body.*

(2) *Put into the scale-pans weights so chosen that the beam remains horizontal.*

(3) *Interchange these weights.*

If the beam still remain horizontal, it is certain that the balance is just.

73. Besides being just, a balance ought to possess great *sensibility*, that is, when the beam remains horizontal, after the scale-pans have equal weights placed in them, a small additional weight ought immediately to deflect the beam visibly from its horizontal position, and this deflection should be the same for the same weight, whatever be the equal weights placed in the two scale-pans. That this may be so, the balance must satisfy the two following conditions of sensibility:

(1) *The fulcrum and the points of suspension of the scale-pans must be in a straight line.*

(2) *The centre of gravity of the beam must be below the fulcrum and very near it.*

For when the first of these conditions is satisfied, then, whatever be the equal weights placed in the two scale-pans, the scale-pans so loaded will be two equal forces applied at the two points of suspension (A and B); these two forces will have a resultant passing through the fulcrum (C), and this resultant will therefore be destroyed by the reaction of the fixed point C, in every position of AB. The beam will therefore be under the same conditions as if no scale-pans were suspended from its extremities, and it will consequently take a horizontal position merely from the action of its own weight applied at its centre of gravity. A difference of 1 oz., for example, will produce the same effect as if the beam were subjected to a force of 1 oz. applied to its extremity A. Hence it is clear that the same difference between the weights of the bodies which are put in the two scale-pans will always produce the same inclination of the beam, whatever be these weights, and that this inclination will be greater, the nearer the centre of gravity of the beam is to the fulcrum.

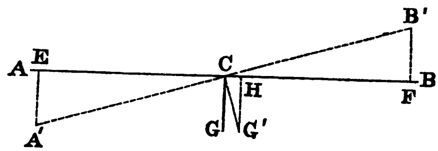
When a balance possesses great sensibility, the addition of a very small weight in one of the scale-pans deflects it from its position of equilibrium, but it stops at another position of equilibrium after having performed a series of oscillations on each side of this new position of equilibrium.

In order that we may not have to wait till the oscillations cease, there is fixed to the beam an index which oscillates with the beam, and the extremity of which moves along an arc DEF of a graduated circle: when the index is observed to oscillate to the same distance on each side of the point (E) of this arc, which corresponds to the horizontal position of the beam, it is certain that the weights put in the two scales are equal, and it is not necessary to wait until the oscillations cease, for discovering whether the beam be horizontal.

74. A good balance should also be *stable*, or possess *stability;* that is, if the equilibrium be slightly disturbed, the balance should readily return to its original position of equilibrium.

75. *To determine the position of equilibrium of a balance when loaded with unequal weights, and to investigate the conditions of* sensibility *and* stability.

Let AB be the beam in a horizontal position, C the



fulcrum, and G its centre of gravity.

Let $A'B'$ be the position of equilibrium of the beam, and G' the position of its centre of gravity, when the weight $W_1 + W_2$ is in the scale-pan suspended from A', and W_1 in that suspended from B'.

Let the angle $ACA' = \theta$, $CG = h$, $W =$ the weight of the beam, and $2a =$ its length.

Let the verticals through A', B', and G' meet the horizontal line through C in E, F, H respectively.

Since the resultant of the two equal weights W_1, W_1 is destroyed by the reaction of the fixed point C, we have, by taking moments about C,

$$W_2 . CE = W . CH,$$

or
$$W_2 . a \cos\theta = W . h \sin\theta;$$

therefore
$$\tan\theta = \frac{W_2}{W} . \frac{a}{h} \quad \text{.......................} \quad (1).$$

Hence, for the same difference (W_2) of the weights, $\tan\theta$

varies as $\frac{a}{h}$, and therefore θ increases or decreases as $\frac{a}{h}$ increases or decreases.

The sensibility will therefore be increased by increasing a *or diminishing* h, *or by increasing the fraction* $\frac{\mathrm{a}}{\mathrm{h}}$.

Again, when the weights are equal, the moment about C of the force tending to bring back the balance from the new position $A'B'$ to its original position AB, is

$$W.CH, \text{ or } W.h\sin\theta, \text{ and this increases with } h.$$

Hence, *the larger* h *is, the greater is the stability of the balance.*

We see then that by increasing both h and a, but a in a greater ratio, the requisites of *sensibility* and *stability* can both be secured.

76. *To determine the true weight of a body by a false balance.*

[By a "false balance" is here meant a balance with unequal arms, which remains horizontal when no bodies are placed in the scale-pans.]

Let a and b be the unknown lengths of the arms of the balance, and W the true weight of the body.

Let W, when suspended from the arm a, be balanced by W_1, suspended from the arm b; and when W is suspended from the arm b, let it be balanced by W_2, suspended from the arm a; then taking moments about the fulcrum, for these two cases of equilibrium, we have

$$W.a = W_1.b,$$

$$W.b = W_2.a,$$

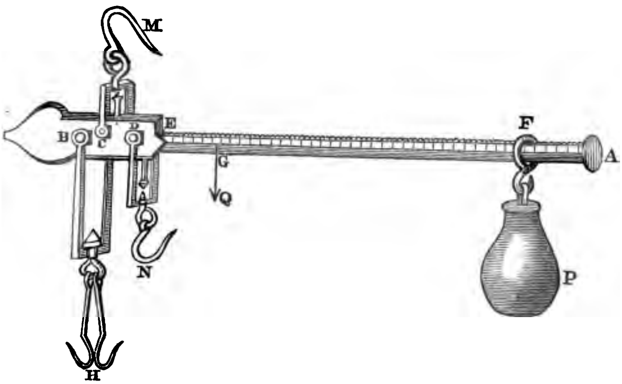
therefore $$W^2.ab = W_1 W_2.ab;$$

and therefore $$W = \sqrt{(W_1 W_2)}.$$

Hence, *the true weight is a mean proportional between the apparent weights.*

77. THE ROMAN OR COMMON STEELYARD.

The *Common Steelyard* consists of a lever AB suspended from a point C, about which it can turn freely.



To the point B are attached two hooks H, (or sometimes only one hook or a scale-pan is attached to B), from which the body to be weighed is suspended.

A ring F, capable of sliding along AE, supports a constant weight P called the *Ball.* AE is graduated, and when a body W is suspended from H, P is moved along AE until AB remains horizontal. The graduation at which P rests determines the weight of W.

Many balances have two rings, (as M and N in the figure) by either of which the instrument may be suspended. In this case the hooks H can turn round the extremity of the lever, in order that they may always hang downwards.

AE is generally a steel prism on a square base (that is, a section of AE by a plane perpendicular to its length is a square), and when the instrument is suspended from C or D, one edge of this prism is turned upwards. Two opposite faces of AE are graduated, corresponding to the two rings M and N.

When the instrument is suspended from C, it is clear that heavier bodies can be weighed than when it is suspended from D.

AE may be graduated *experimentally* by marking at the points where P stops, when it balances a series of known weights suspended from H, the numbers representing these weights; or we may graduate AE by the method explained in the following proposition.

78. *To graduate the Common Steelyard.*

(Fig. Art. 77.)

Let the steelyard be suspended from C, and let Q and G be the weight and centre of gravity of the instrument, exclusive of P and M.

If P at F balance W at H, by taking moments about C, we have

$$Q.CG + P.CF = W.CB \qquad (1).$$

Take a point O on EB such that $P.CO = Q.CG$, then (1) becomes $P.CO + P.CF$, or $P.OF = W.CB$,

therefore $$OF = \frac{W}{P}.CB \qquad (2).$$

If we make W successively equal to P, $2P$, $3P$, &c., then OF will be equal to CB, $2CB$, $3CB$, &c.

Hence, we may graduate AB by taking distances from O successively equal to CB, $2CB$, $3CB$ &c., and marking 1, 2, 3, &c. at their extremities. These divisions may be subdivided. When P rests at 1, 2, 3 &c., the weights of the body at H will be successively equal to P, $2P$, $3P$, &c., and these weights are known, since P is known.

79. THE DANISH STEELYAND.

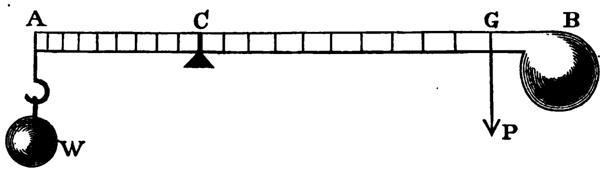
The Danish Steelyard consists of a lever terminating in a ball at one end; the body to be weighed is suspended from the other end, and the fulcrum is moveable.

[This instrument is now never used, so far as I have been able to ascertain.

The Pillar-Stand Scales described in Art. 72, and the Steelyard in Art. 77, are of the best modern construction, and have been selected for description after a careful examination of a great variety of balances.]

80. *To graduate the Danish Steelyard.*

Let P be the weight of the instrument, G its centre of



gravity, W the weight of a body suspended from A, and C the position of the fulcrum when P and W balance about it; then taking moments about C,

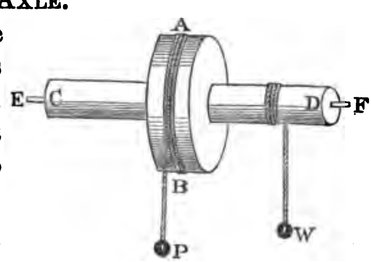
we have $$W.AC=P.CG=P(AG-AC),$$

therefore $$AC=\frac{P}{P+W}.AG$$

by making W successively equal to P, $2P$, $3P$, &c., the successive values of AC are determined, and these values determine the graduations.

81. The Wheel and Axle.

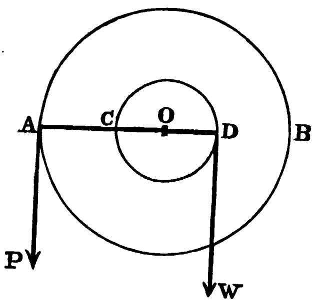
The wheel (AB) and axle (CD) consists of two cylinders rigidly connected together, and having a common axis which terminates in two pivots (E, F).



The machine turns round

these pivots which rest in fixed sockets. A cord coiled round the wheel has one end fastened to the wheel, and at the other end the power (P) is applied. Another cord coiled round the axle in the opposite direction has one end fastened to the axle, and the other attached to the weight (W).

82. *To find the condition of equilibrium of the Wheel and Axle.*



P and W will exert the same efforts to turn the machine about its axis in whatever planes perpendicular to the axis they may act. Suppose then P and W to act in the same plane, and let the plane through their lines of action meet the axis of the machine in O, the wheel in the circle AB, and the axle in the circle CD.

Since there is equilibrium, we may suppose the cords to be rigidly attached to the machine at the points A and D, where they quit it;

hence, taking moments about O, we have

$$P.OA = W.OD,$$

or
$$\frac{P}{W} = \frac{OD}{OA} = \frac{\text{radius of axle}}{\text{radius of wheel}}.$$

Hence, *when there is equilibrium on the wheel and axle,* P *is to* W *as the radius of the axle is to the radius of the wheel.*

The converse is manifestly true, namely, *If* P *be to* W *as the radius of the axle to the radius of the wheel, there will be equilibrium.*

For P and W are two parallel forces, and therefore their resultant will (Art. 28) pass through O and be destroyed by the reaction of this fixed point.

[Or thus. By hypothesis, $P:W::OD:OA$; therefore $P.OA = W.OD$, therefore the resultant passes through O, since it cannot vanish.]

83. THE PULLEY.

A *Pulley* is a uniform circular disc or wheel of hard wood or metal, turning freely on an axis through its centre at right angles to its plane, and having a groove cut on its circumference. It is usually enclosed in a frame called the *block*. The axis may be fixed to the pulley, and then its two extremities turn in two circular apertures formed in the block, or else the axis may be fixed to the block, and pass through a circular aperture pierced at the centre of the pulley. In this case the pulley turns without carrying the axis with it. A cord passes round a portion of the groove, and quits it on both sides in the direction of tangents to its circumference.

A pulley is called *fixed* or *moveable* according as its block is fixed or moveable.

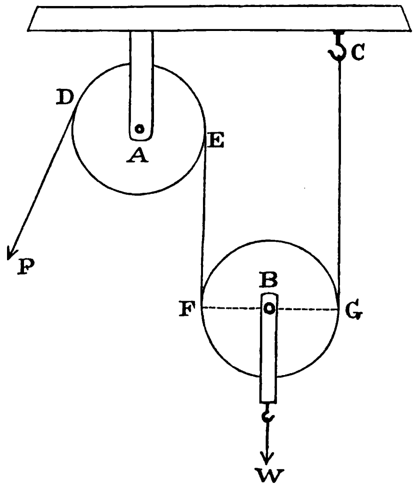
An assemblage of several pulleys is called a *system of pulleys*.

N.B. In the following investigations we shall assume the cords to be without appreciable weight or thickness. We shall also assume the cords to be perfectly flexible and inextensible, and the pulleys and their axes to be perfectly smooth.

The portions of the cords not in contact with the pulleys are parallel in the next four propositions.

84. *To find the condition of equilibrium in the single moveable pulley.*

Let B be the centre of the single moveable pulley, w_1 its



weight, F and G the points where the cord $DEFGC$, passing round the pulley, quits it.

Since the cord $DEFGC$ is perfectly flexible, and the pulleys A and B are smooth, its tension is the same throughout, and equal to P.

Also, since there is equilibrium, we may suppose the cord to be rigidly attached to the pulley at F and G, without disturbing the equilibrium.

The pulley may therefore be regarded as a rigid body maintained in equilibrium by the two tensions in FE and GC, each equal to P, and acting vertically upwards, and the weight $W + w_1$ acting vertically downwards through B.

Hence $$2P = W + w_1 \quad\text{.......................(1).}$$

Q. E. F.

If we neglect the weight of the pulley, we have

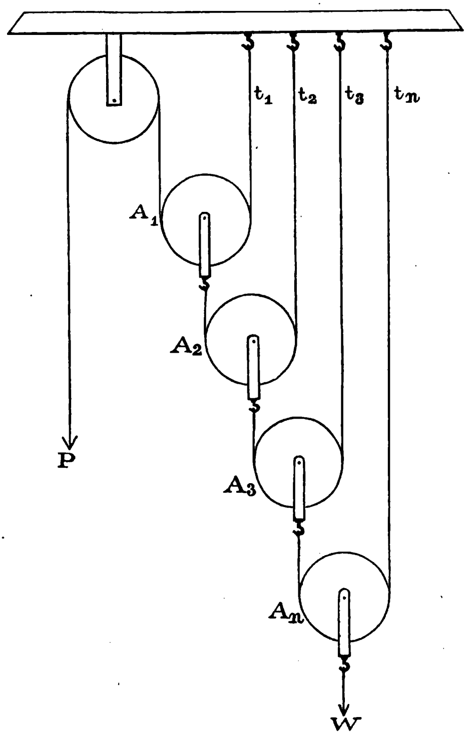
$$2P = W,$$

$$\text{or } \frac{W}{P} = \frac{1}{2} \quad\text{............................(2).}$$

It will be observed that the only effect of the fixed pulley A is to change the direction of the power (P).

85. *To find the condition of equilibrium in the* first system of pulleys *in which the cord passing round any pulley has one end fixed, and the other attached to the pulley next above it, or to the power.*

Let $w_1, w_2, w_3, \ldots w_n$ be the weights of the pulleys $A_1, A_2,$



$A_3, \ldots A_n$ respectively, and $t_1, t_2, t_3, \ldots t_n$ the tensions of the cords passing round these pulleys; then the pulley A_1 is acted on by the two tensions t_1 vertically upwards, the tension t_2 and its own weight w_1 vertically downwards; therefore for its equilibrium we have

$$t_2 + w_1 = 2t_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly, for the equilibrium of the other pulleys we have

$$t_3 + w_2 = 2t_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$t_4 + w_3 = 2t_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$
$$\cdots\cdots$$
$$t_n + w_{n-1} = 2t_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(n-1),$$
$$W + w_n = 2t_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n).$$

And it is evident that

$$t_1 = P.$$

From this equation and (1) we have

$$t_2 = 2P - W_1.$$

Again, from this and (2) we have

$$t_3 = 2^2P - 2W_1 - w_2.$$

Similarly, $t_4 = 2^3P - 2^2w_1 - 2w_2 - w_3$,

$$\cdots\cdots\cdots\cdots\cdots$$
$$t_n = 2^{n-1}P - 2^{n-2}w_1 - 2^{n-3}w_2 \ldots - w_{n-1},$$
$$W = 2^nP - 2^{n-1}w_1 - 2^{n-2}w_2 \ldots - 2w_{n-1} - w_n,$$

or, $2^nP = W + 2^{n-1}w_1 + 2^{n-2}w_2 + 2^{n-3}w_3 + \ldots + 2w_{n-1} + w_n \ldots(\alpha)$,

the required condition of equilibrium. Q. E. F.

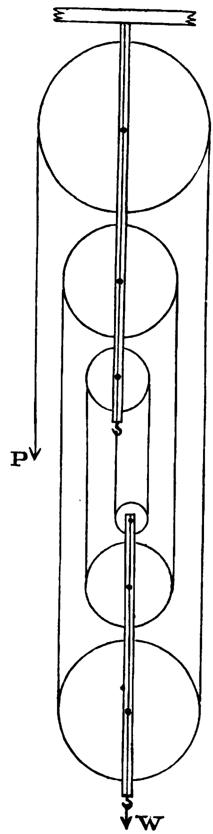
If the weight of each pulley be equal to w, then (α) becomes $2^nP = W + (2^{n-1} + 2^{n-2} + \ldots + 2 + 1)w = W + (2^n - 1)w \ldots(\beta)$.

If the weight of the pulleys be neglected, then

$$2^n . P = W,$$

or
$$\frac{P}{W} = \frac{1}{2^n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

86. *To find the condition of equilibrium in the* second system of pulleys *in which there are two blocks*—the one moveable and the other fixed—*and the same cord passes round all the pulleys.*



Let n be the number of parallel portions of the cord at the lower block, and W_1 the weight of the block.

Since the tension of the cord is the same throughout and equal to P, therefore the resultant tension upon the lower block is nP acting vertically upwards, and this must equilibrate $W + W_1$ which acts vertically downwards,

therefore $$nP = W + W_1 \quad \text{.........(1)}$$

the required condition of equilibrium.

Q. E. F.

If the weight of the lower block be neglected,

$$nP = W,$$

or, $$\frac{P}{W} = \frac{1}{n} \quad \text{.......................(2)}.$$

87. *To find the condition of equilibrium in the* third system of pulleys *in which the cord passing round any pulley has one end attached to the weight, and the other to the pulley next below it, or to the power.*

Let there be n pulleys, the highest being fixed.

Let $w_1, w_2, \ldots w_{n-1}$ be the weights of the pulleys $A_1, A_2 \ldots A_{n-1}$ respectively, and $t_1, t_2, \ldots t_n$ the tensions of the cords passing round the pulleys.

For the equilibrium of the pulleys $A_1, A_2, \ldots A_{n-1}$ in order, we have

$t_2 = 2t_1 + w_1$ ....(1),

$t_3 = 2t_2 + w_2$ ....(2),

$t_4 = 2t_3 + w_3$ ....(3),

. . . .

$t_n = 2t_{n-1} + w_{n-1}$ ....$(n-1)$,

and clearly $t_1 = P$;

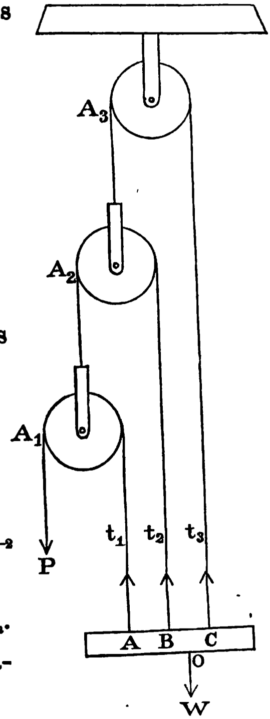
therefore from the above equations we have

$t_2 = 2P + w_1$,

$t_3 = 2^2P + 2w_1 + w_2$,

. . . .

$t_n = 2^{n-1}P + 2^{n-2}w_1 + 2^{n-3}w_2 + \ldots + 2w_{n-2} + w_{n-1}$.



Also evidently $W = t_1 + t_2 + \ldots + t_n$.

Hence, by adding the last n equations we have

$$\begin{aligned} W = &(1 + 2 + 2^2 + \ldots + 2^{n-1})P \\ &+ (1 + 2 + 2^2 + \ldots + 2^{n-2})w_1 \\ &+ (1 + 2 + 2^2 + \ldots + 2^{n-3})w_2 \\ &+ \ldots\ldots\ldots\ldots\ldots \\ &+ w_{n-1} \\ = &(2^n - 1)P + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \ldots + (2^2 - 1)w_{n-2} \\ &+ (2-1)w_{n-1} \ldots\ldots(\alpha), \end{aligned}$$

the required condition of equilibrium. Q. E. F.

If the weight of the pulleys be neglected, then

$$W = (2^n - 1)P,$$

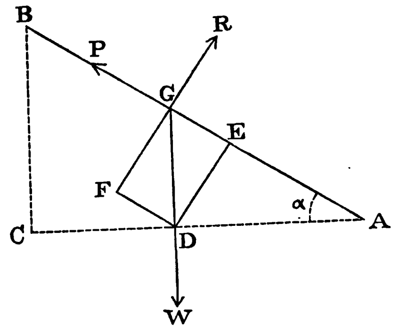
or,

$$\frac{P}{W} = \frac{1}{2^n - 1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

88. THE INCLINED PLANE. An *inclined plane* is a plane inclined to a horizontal plane at any angle. This angle is called the *inclination* of the plane.

Let (see fig. Art. 89) a vertical plane perpendicular to the inclined plane intersect it in the straight line AB, and the horizontal plane in AC, and draw BC vertical; then AB is called the *length* of the inclined plane, BC its *height*, and CA its *base*.

89. (*a*) *A body whose weight is* W *is maintained in equilibrium on a smooth inclined plane by a force* P *acting up the plane; to find the conditions of equilibrium.*



Let G be the position of the body on the inclined plane AB, and R the reaction of the plane on the body.

Since the plane is smooth the direction of R will be normal to the plane.

Draw the vertical GD meeting the base of the plane in D, and complete the rectangle FE.

The body is maintained in equilibrium by the three forces P, W, R, and the directions of these forces are respectively parallel to the sides of the triangle EGD; therefore, by the triangle of forces,

$$P : W : R :: EG : GD : DE;$$

but the triangles EGD and ABC are clearly equiangular,

therefore $EG : GD : DE :: CB : BA : AC$,

and therefore $P : W : R :: CB : BA : AC$,

or, $$\frac{P}{CB} = \frac{W}{BA} = \frac{R}{AC} \text{ .... } (1),$$

the required conditions of equilibrium. Q. E. F.

We may enunciate the first equation of (1) as follows:—

If P *act up the plane, and there be equilibrium, then* P *is to* W *as the height of the plane is to its length.*

Conversely, (β) *If* P *act up the plane and* P : W :: CB : BA, *then the body will be in equilibrium.*

The same construction being made, since the triangles DGF and ABC are equiangular,

therefore $$GD : DF :: AB : BC$$
$$= W : P.$$

Hence we may represent W and P by GD and DF, as these lines are parallel to the directions of W and P and proportional to them; therefore (Art. 17) GF must represent the resultant of W and P in direction and magnitude. Hence the resultant of W and P is normal to the plane, and will therefore be destroyed by its reaction.

The body will therefore remain in equilibrium on the plane. Q. E. D.

*(α) Otherwise thus. Let $\alpha = BAC$ the inclination of the plane; then resolving the forces P, W, R, which act on the body, along and perpendicular to the plane, we have, for equilibrium,

$$\left.\begin{array}{l} P = W\sin\alpha \\ R = W\cos\alpha \end{array}\right\} \text{..........................} (2),$$

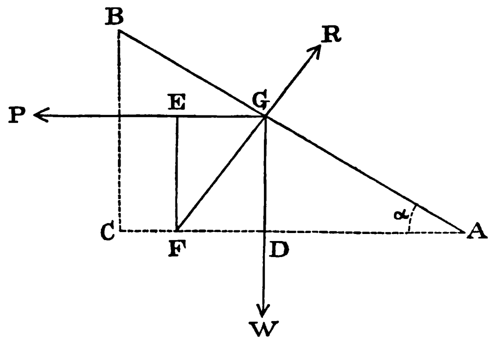
the required conditions of equilibrium.

[illegible] (α) *A body whose weight is* W *is maintained in* [illegible]*brium on a smooth inclined plane by a force* P *acting* [illegible]*ntally; to find the conditions of equilibrium.*

[illegible] G be the position of the body on the inclined plane [illegible] the reaction of the plane on the body.

[illegible] plane is smooth the direction of R will be normal

Draw GF perpendicular to AB and meeting the base of



the plane in F, and complete the rectangle DE.

The body is maintained in equilibrium by the three forces W, P, R, and the directions of these forces are respectively parallel to the sides of the triangle GDF; therefore, by the triangle of forces,

$$P : W : R :: FD : DG : GF;$$

but the triangles GDF and ABC are clearly equiangular,

therefore $\qquad FD : DG : GF :: BC : CA : AB,$

and therefore $\qquad P : W : R :: BC : CA : AB,$

or, $$\frac{P}{BC} = \frac{W}{CA} = \frac{R}{AB} \quad \text{.......................... (1),}$$

the required conditions of equilibrium. Q. E. F.

We may enunciate the first equation of (1) as follows:

If P *act horizontally and there be equilibrium, then* P *is to* W *as the height of the plane to its base.*

Conversely, (β) *If* P *act horizontally and* P : W :: BC : CA, *then the body will be in equilibrium.*

The same construction being made, since the triangles GDF and ABC are equiangular, therefore $GD : DF :: AC : CB$

$= W : P.$

Hence we may represent W and P by GD and DF, as these lines are parallel to the directions of W and P, and proportional to them; therefore (Art. 17) GF must represent the resultant of W and P in direction and magnitude. Hence the resultant of W and P is normal to the plane, and will therefore be destroyed by its reaction.

The body will therefore remain in equilibrium on the plane. Q. E. D.

*(α) Otherwise thus. Resolving the forces P, W, R, which act on the body, along and perpendicular to the plane, we have, for equilibrium,

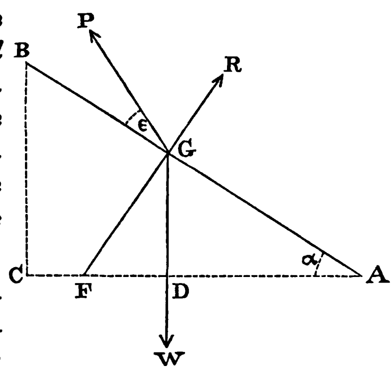
$$P\cos\alpha = W\sin\alpha, \text{ or } P = W\tan\alpha,$$

$$R = P\sin\alpha + W\cos\alpha = W\tan\alpha \,.\, \sin\alpha + W\cos\alpha = \frac{W}{\cos\alpha};$$

therefore and
$$\left.\begin{aligned} P &= W\tan\alpha \\ R &= W\sec\alpha \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (2)$$

are the required conditions of equilibrium.

*91. *A body whose weight is* W *is maintained in equilibrium on a smooth inclined plane by a force* P, *the direction of which makes an angle* ε *with the plane; to find the conditions of equilibrium.*



Let G be the position of the body on the inclined plane AB, and R the reaction of the plane on the body; then resolving the forces

P, W, R, which act on the body, along and perpendicular to the plane, we have, for equilibrium,

$$P\cos\epsilon - W\sin\alpha = 0, \text{ or } P = W.\frac{\sin\alpha}{\cos\epsilon},$$
$$R + P\sin\epsilon - W\cos\alpha = 0;$$
$$\therefore \quad R = W\cos\alpha - P\sin\epsilon = W\cos\alpha - W.\frac{\sin\alpha}{\cos\epsilon}.\sin\epsilon$$
$$= W.\frac{\cos(\alpha+\epsilon)}{\cos\epsilon}.$$

$$\left.\begin{aligned} \text{Hence,}\quad & P = W.\frac{\sin\alpha}{\cos\epsilon} \\ \text{and}\quad & R = W.\frac{\cos(\alpha+\epsilon)}{\cos\epsilon} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

are the required conditions of equilibrium. Q. E. F.

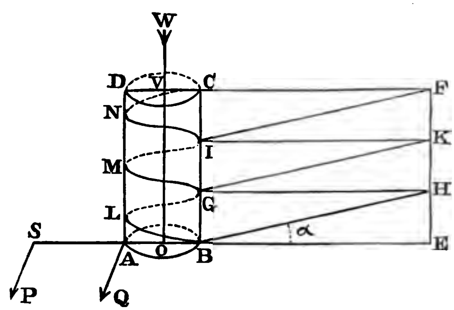
92. THE SCREW.

A *screw* is a uniform projecting thread or rib winding round the curved surface of a solid cylinder, and everywhere inclined at the same angle to the base of the cylinder. This angle is called the *pitch* of the screw. The solid cylinder fits into a cylindrical aperture pierced in a block, and having a groove cut into its surface in which the projecting thread works.

The weight acts in the direction of the axis of the cylinder, and the power is applied perpendicularly at the extremity of a straight lever or bar which is attached to the cylinder at right angles to its axis.

Since the pitch of the screw is everywhere the same, we may regard the screw as formed by wrapping a series of equal inclined planes round a cylinder in the following manner.

Let $ABCD$ be a solid cylinder, and $BEFC$ a rectangle whose base BE is equal to the circumference of the cylinder. Divide the rectangle EC into a number of equal rectangles by lines parallel to BE, as in the figure, and draw their diagonals



BH, GK, IF. If now the rectangle EC be wrapped upon the cylinder so that BE coincides with the circumference of the base, the points E, H, K, F, will respectively fall upon the points B, G, I, C, of the cylinder, and the lines BH, GK, IF, will trace out on its surface the continuous spiral thread $BLGMINC$ winding round the cylinder, and everywhere inclined at the same angle to its base. If we now suppose this thread to become protuberant, we shall have the screw traced on the cylinder.

93. *To find the ratio of* P *to* W *in the screw, when they are in equilibrium.*

(Fig. Art. 92.)

Let α be the pitch of the screw, W the weight acting in the direction of VO the axis of the solid cylinder, and P the power acting at S perpendicularly to OS.

Let $OA = r$, and $OS = a$.

Also let Q be the force which, if applied at the circumference of the cylinder at right angles to OA, would be equivalent to P applied at S, so that $Q.AO = P.SO$;

$$\therefore \quad Q = P.\frac{SO}{AO} = P.\frac{a}{r} \quad \text{..................(1).}$$

Now the force W is equilibrated by Q and the reactions of the threads of the screw. Since the thread is smooth and has everywhere the same pitch, these reactions are parallel, and have therefore a single resultant (R). The conditions of equilibrium are therefore the same as on an inclined plane whose inclination is α, when the power acts horizontally. Hence (Art. 90),

$$\frac{Q}{W}=\frac{\text{height of plane}}{\text{base}}=\frac{HE}{BE}=\frac{HE}{2\pi r},$$

or by (1)

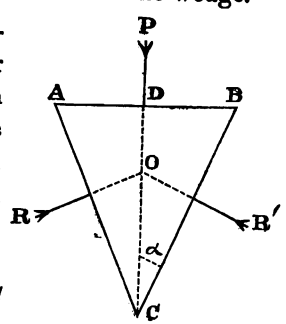
$$\frac{P}{W}\cdot\frac{a}{r}=\frac{HE}{2\pi r};$$

$$\therefore \frac{P}{W}=\frac{HE}{2\pi a}=\frac{\text{distance between two threads}}{\text{circumference of circle described by power}}.$$

Q. E. F.

*94. The Wedge.

A *wedge* is a solid triangular prism, principally used for splitting timber or for raising heavy bodies through a small space. We shall only consider the case of the equilibrium of an isosceles wedge. The *ends* of such a wedge are similar and equal isosceles triangles; its two *faces* are similar and equal rectangles, which meet in the *edge* of the wedge; and the remaining rectangle is called the *head* or *back* of the wedge.



Let ABC be a section of an isosceles wedge by a plane perpendicular to its edge and meeting the edge in C, the faces in AC and BC, and the back in AB, so that ACB is an isosceles triangle equal and parallel to the two ends of the wedge.

Let the power P be applied to the back of the wedge in the direction DC which bisects the angle ACB (2α).

Since the wedge is assumed to be smooth, the two resistances (R and R') must act perpendicularly to the faces.

Hence, resolving the forces which act on the wedge along and perpendicular to DC, we have for equilibrium,

$$P = R\sin\alpha + R'\sin\alpha,$$
$$R\cos\alpha = R'\cos\alpha;$$

$\therefore\ R = R'$, and therefore $P = 2R\sin\alpha$.

Also the three forces must pass through the same point.

These are the three conditions of equilibrium of an isosceles wedge. Q. E. F.

Since $\sin\alpha = \frac{BD}{BC}$, therefore $P = 2R \cdot \frac{BD}{BC}$, or $\frac{P}{R} = \frac{AB}{BC}$.

This result is only a particular case of the following proposition, and both may be easily proved without the aid of Trigonometry.

If three forces acting perpendicularly to the sides of a triangle be in equilibrium, they pass through the same point, and are proportional to the sides to which they are respectively perpendicular.

CHAPTER VII.

ON FRICTION.

95. All bodies with which we are acquainted offer a resistance to motion when we attempt to make the surface of one body slide over that of another. This resistance is called *friction*, and is found by experiment to be different in different bodies. The surfaces of such bodies are said to be *rough*. We have hitherto *assumed* the surfaces of bodies to be perfectly smooth, and consequently the results obtained on this hypothesis will only be first approximations to the real state of the case.

The direction of the mutual action between smooth surfaces in contact is always in the common normal to the surfaces, but the direction of the mutual action between rough surfaces in contact depends upon the amount of friction as well as upon the normal pressure.

When the plane surface of one body is pressed *obliquely* against that of another body, and the first body is on the point of sliding, it is said to be in a state *bordering on motion*, and the greatest amount of friction is then called into play, in other words, the friction *at starting* is a maximum.

96. It is found by experiment that when one body is sliding over the surface of another,

The friction during motion is

1. *Proportional to the normal pressure,*

2. *Independent of the extent of surfaces in contact,*
3. *Independent of the velocity of the motion.*

Again, it is found that when the bodies have been some time in contact,

The friction at starting is also

1. *Proportional to the normal pressure,*
2. *Independent of the extent of surfaces in contact;* but that for compressible bodies it is considerably greater than the friction during the motion.

The first three laws may be called *the laws of Dynamical friction,* the latter two *the laws of Statical friction.*

97. When the friction is assumed to be so great as to prevent sliding, whatever be the normal pressure, the surfaces are said to be *perfectly rough.*

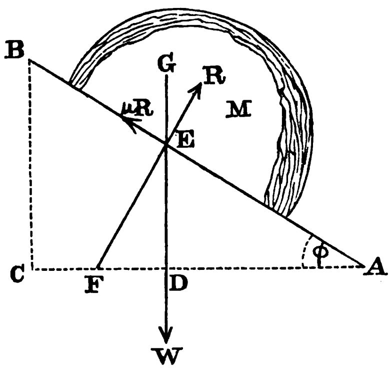
If we denote the normal pressure by R and the maximum friction by F, then the ratio $\frac{F}{R}$ is called the *coefficient of friction,* and is commonly denoted by μ. For perfectly rough surfaces $\mu = \infty$. In any statical problem the maximum amount of friction is only called into play when one body is in the state bordering on motion. The friction always acts in the plane of contact, and in the direction opposite to that in which relative motion would begin if the surfaces were smooth.

Friction cannot produce motion in any body, but it can destroy motion. It may therefore be called a *destroying* force. It is thus distinguished from gravity and such forces as can produce motion, and which may therefore be called *moving* forces.

98. THE ANGLE OF FRICTION.

If a body subject only to the action of gravity rest upon a horizontal plane, and the plane be gradually turned about a horizontal line until the body is in a state bordering on motion, then the inclination of the plane is called the *angle of friction.*

Let AB be the position of the inclined plane when the



body M, whose weight is W, is just on the point of sliding down the plane, F the friction, and R the reaction of the plane on the body; then, resolving the forces which act on M along and perpendicular to the plane, we must have, for equilibrium,

$$\left.\begin{array}{l} R = W\cos\phi \\ F = W\sin\phi \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

but, by the first law of friction, we have also $F = \mu R$;

hence $$W\sin\phi = \mu W\cos\phi,$$

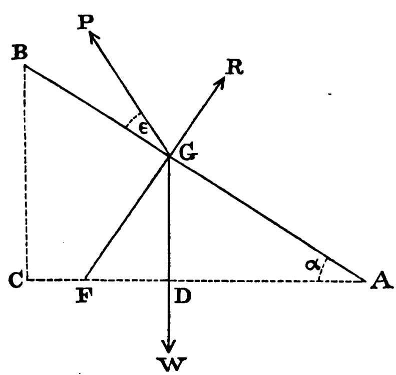
therefore $$\tan\phi = \mu \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (2) we see that *the tangent of the angle of friction is equal to the coefficient of friction.*

99. (Fig. Art. 98).

If a force P acting in the direction GE press a body which is subjected to no other force, against a plane AB, it can be shown, as in the last article, that when the body is in a state bordering on motion the angle GER is equal to the angle of friction. Also, if a right cone be described having RE for its axis and GER for its semivertical angle, then, if P act in any direction within this cone, the body will remain at rest, for the resolved part of P along the plane will be less than the maximum friction; but if the direction of P lie without the cone, then its resolved part along the plane will exceed the maximum friction, and the body will therefore begin to slide along the plane. Hence the angle ϕ is sometimes called *the limiting angle of friction.*

100. *A body whose weight is* W *is maintained in equilibrium on a rough inclined plane by a force* P, *the direction of which makes an angle* ϵ *with the plane; to find the values of* P (1) *when the body is on the point of moving* down *the plane,* (2) *when it is on the point of moving* up *the plane.*



(1) Let R be the reaction of the plane on the body, P_1 the

required value of P in this case, and μ the known coefficient of friction; then μR the friction acts *up* the plane.

Therefore, resolving the forces acting on the body along and perpendicular to the plane, we have for equilibrium,

$$\left.\begin{aligned} P_1 \cos\epsilon + \mu R - W\sin\alpha = 0 \\ R + P_1 \sin\epsilon - W\cos\alpha = 0 \end{aligned}\right\},$$

$$\therefore\ P_1 = W.\frac{\sin\alpha - \mu\cos\alpha}{\cos\epsilon - \mu\sin\epsilon} = W.\frac{\sin(\alpha-\phi)}{\cos(\epsilon+\phi)} \text{ .... } (\alpha),$$

by putting $\tan\phi$ for μ.

(2) If P_2 be the value of P in this case, we shall find similarly, or by writing $-\mu$ for μ in (α),

$$P_2 = W.\frac{\sin\alpha + \mu\cos\alpha}{\cos\epsilon + \mu\sin\epsilon} = W.\frac{\sin(\alpha+\phi)}{\cos(\epsilon-\phi)} \text{ .... } (\beta).$$

Q. E. F.

[The laws of friction enunciated in the present Chapter have been derived from General Morin's work, entitled *Leçons De Mécanique Pratique*, 2nd edit., Paris, 1855, pp. 253—315.

In these 63 pages the reader will find a very clear account of the refined contrivances and experiments for determining the amount and laws of friction.]

EXAMPLES ON CHAPTERS I. AND II., Pp. 1—6.

1. If we select an inch and a lb. as units of length and weight respectively, what force would be represented by a line of ·732 feet?

2. What is the length of the line representing a force of ·54 oz., a foot and a lb. being the units?

3. Two forces of 18 lbs. and 24 lbs. act in directions at right angles to each other, upon a point of a body; find the magnitude of their resultant.

4. Two forces acting at right angles have a resultant of 14 lbs. One of the components is 7 lbs.; find the other component and the angle it makes with the resultant.

5. The resultant of two equal forces acting at a point at an angle of 90° is 10 lbs.; find the components.

6. If the resultant be at right angles to one component and be equal to half the other, what is the angle between the components?

7. Two forces of 15 and 22 lbs. act upon a point at an angle of 60°; required the resultant. If 120° were the angle of inclination, what would be the resultant?

8. The resultant of two forces acting at 45° is 9 lbs., one of the components is 5 lbs.; find the other, and its inclination to the resultant.

9. Three forces act upon a point. Two of them are each equal to 2 lbs., and are inclined at an angle of 120°. The third 3 lbs. acts between them, and is inclined to one of the equal forces at an angle of 15°; find the magnitude of the resultant.

10. Three forces acting on a point are in equilibrium, when their directions are inclined to each other at angles of 150°, 120°, 90°; find the ratios of the forces.

11. Two weights of 3 lbs. and 4 lbs. are connected by a string, which passes over two smooth pegs in the same horizontal line; what weight must be attached to the string between the pegs in order that, when the weights have assumed their position of equilibrium, the string may be bent at right angles.

12. Could three forces in proportion (1) of 13, 17, 21; (2) of 14, 17, 21; (3) of 15, 17, 21, acting upon a point, keep it at rest?

13. Three forces act perpendicularly to the sides of a triangle at the middle points in the order of the sides, and each force is proportional to the side on which it acts. Shew that the forces are in equilibrium. The same proposition is true for any closed polygon.

14. A weight of 17 lbs. is suspended by a string from a fixed point; find the horizontal force which must be applied to the body to keep the string at an inclination of 60° to the vertical.

15. The lower end B of a rigid rod 10 feet long is hinged to an upright post, and its other extremity A is fastened by a string 8 feet long to a point C vertically above B, so that ACB is a right angle. If a weight of 1 ton be suspended from A, what will be the tension of the string?

Examples on Chapter III., Page 21.

1. Two weights of 113 and 252 lbs. are suspended from the extremities of a rigid bar, without weight, 2 feet long; find the resultant and the segments into which it divides the bar.

2. A rigid bar, without weight, and 35 inches long, is supported in a horizontal position by two props at its extremities; find the pressure on each prop when a weight of 2 cwt. 13 lbs. is suspended from a point of the bar, distant 17 inches from one extremity.

3. Two parallel forces of 21 and 37 lbs. act in opposite directions at two points of a rigid bar, without weight, at distance 13 inches from each other; find the resultant and its point of application.

4. A weightless bar is supported in a horizontal position by two props at its extremities. Weights P, Q are suspended from the points of trisection of the rod; find the pressures on the props.

5. ABC is a weightless table in the form of a triangle, and is supported on legs at its corners. A weight is placed at any point O on the table. Shew that the pressure on leg at A is proportional to the area of the triangle BOC.

If the pressures on the legs be equal, O is the intersection of bisectors of sides.

6. A rod AB, without weight, 14 inches long, is suspended by two strings from a peg C, the string AC is 15 and BC 13 inches long; 130 lbs. is suspended from A, and 52 lbs. from B: when the whole is in equilibrium, find the tensions of the strings, the pressure along the rod, and the resultant action on the peg.

EXAMPLES ON CHAPTERS IV. AND V., Pp. 29—39.

1. A uniform beam, whose weight is 1 cwt. 3 lbs., and length 10 feet, has weights of 19 and 23 lbs. hung on at its ends; find the point about which the beam will balance.

2. A uniform beam, of length 8 feet, and weight 183 lbs., has weights of 6, 7, 8, 9 pounds hung on at distances 1, 2, 3, 4 feet respectively from one end; find the distance of the C. G. of the system from this end.

3. The weight of a uniform equilateral triangle is 20 lbs., and the length of each side is 9 inches. At one vertex A a body of 5 lbs. is placed; find the C. G. of the system.

4. At the angles of a square whose side is 20 inches, weights are placed proportional to 1, 3, 5, 7; find the distance of the C. G. from the least weight.

5. The diagonals of a square $ABCD$ intersect in O. If the triangle AOB be taken away, find the distance of the C. G. of the remainder from O, the side of the square being 18 in.

6. A rod of uniform thickness is made up of equal lengths of three substances, the densities of which taken in order are in the proportion of 1 : 2 : 3; find the position of the C. G. of the rod.

7. If on the radius of a circle as diameter another circle be described, find the C. G. of the area between the two circles.

EXAMPLES ON CHAPTERS VI. AND VII., Pp. 55, 83.

1. A bar weighs 5 ounces per inch: find its length when a weight 15 ounces suspended at one end keeps it in equilibrium about a fulcrum at a distance of 4 inches from the other end.

2. The length of an oar is 12 feet, and the rowlock $2\frac{1}{2}$ feet from the handle. Compare the force exerted by the rower with the resistance of the boat.

3. The arms of a bent lever, of length 18 inches and 12 inches, are inclined to the horizon at angles of 45° and 30° respectively; if the greater weight be 16 lbs., find the less.

4. If P, Q be the apparent weights of a body when placed in the scales of a false balance, find its true weight W. *e.g.* If the apparent weights be $2\frac{4}{5}$ lbs. and $3\frac{5}{8}$, find the true weight.

5. In a wheel and axle a weight of 555 lbs. is balanced by a power of 10 lbs., the radius of the wheel is 6 feet; find the radius of the axle.

6. A weight of 7 lbs. is supported on an inclined plane by a force of 5 lbs. acting parallel to the plane. The base of the plane is 340 feet; find its height and length.

7. A body of 26 lbs. is placed on a plane inclined at 30° to the horizon; find what force acting along the plane will support it. What is the pressure on the plane?

8. What horizontal force would sustain the weight in the last example?

9. In the fig. of Art. 9, if $\alpha = 30°$, $\epsilon = 45°$, and the weight $= 16$ lbs.; find the pressure on the plane.

10. Find the weight that can be sustained by a power of 1 lb. acting at the distance of 3 yards from the axis of the screw, the distance between two contiguous threads being 1 in.

11. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 150 turns in the height of 12 inches, the length of the arm being 6 feet?

12. In the single moveable pulley (fig. Art. 84) the weight is 13 cwt. 6 lbs.; find the power required to support it.

13. How many pulleys, supposed to be without weight, must there be in the system (fig. Art. 85) in order that 1 lb. may support a weight of 128 lbs.?

14. In the system (fig. Art. 86) if there were 5 pulleys at the lower block, find what power would support a weight of 1000 lbs.

15. In a system of pulleys having four distinct cords (fig. Art. 87) determine the weight supported and the strain upon the fixed pulley, the power being 100 lbs., and the weight of each pulley 5 lbs.

16. A body whose weight is W rests upon a plane inclined to the horizon at an angle of 30°, and the coefficient of friction between the plane and the body is $\frac{1}{2}$. Find in what direction a force equal to W must act, in order just to support the weight.

17. A weight rests upon a rough plane whose inclination is 45°, a force equal to the weight and inclined to the plane at 30° is just about to pull the body up the plane. Find the coefficient of friction.

MISCELLANEOUS PROBLEMS.

1. A beam rests between two smooth planes inclined at angles α, β to the horizon; find the position of equilibrium, and the pressures on the planes.

N.B.—The following Trigonometrical formulæ are useful in finding the positions of equilibrium of beams.

Through the vertex C of the triangle ABC a line is drawn cutting AB in D, and making angles α, β with AC and BC. The angle $CDB = \theta$, and $AD : BD :: m : n$.

Then
$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot A - m\cot B.$$

2. If the planes be rough, the angle of friction being ϕ, find the limiting position of equilibrium, the beam being on the point of slipping up the plane whose inclination is α.

3. Find the position of equilibrium of a uniform beam of length $2a$, with one end in a hemispherical bowl of radius r, a being greater than r.

4. A beam rests upon a smooth peg with one extremity A against a smooth vertical wall; find the position of equilibrium.

5. A beam rests with one end upon the ground and the other against a wall; a point in the beam is attached by a string to the foot of the wall: find the tension of the string.

6. A heavy triangle is hung up by the angle C, and the opposite side is inclined at an angle γ to the horizon; shew that

$$2\tan\gamma = \cot A - \cot B.$$

7. The altitude of a right cone is h, and the radius of the base is r; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg. Shew that if the cone rests with its axis horizontal, the length of the string is $\sqrt{(h^2 + 4r^2)}$.

8. Three equal forces act at one point; α, β, γ are the angles between their directions, so that $\alpha + \beta + \gamma = 2\pi$; shew that their resultant bears to any one of them the ratio

$$\left(1 - 8\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)^{\frac{1}{2}} : 1.$$

9. A circular cylinder is supported between two perfectly rough inclined planes sloping the same way by means of a cord wound round it. The cord passes over a pulley in the upper plane, and has a weight P attached to its free end. Find the limits between which P must lie for equilibrium.

10. Find the C. G. of a trapezium whose parallel sides are a, b.

11. Two weights P, Q support each other on two planes inclined at angles α, β to the horizon, by means of a string passing over the common vertex of the planes. Find the relation between P and Q, and the tension of the string.

12. The extremities M and N of a string are fixed, and to given points A, B, C ... in it, weights P, Q, R ... are attached; find the tensions of the several parts of the string in terms of their inclinations to the vertical and the weights P, Q, R ....

13. A beam OP turns about a hinge at O, and touches at P a cylinder which rests upon the ground at R, the coefficients of friction of the surfaces at P and R being the same. The friction is supposed gradually to diminish until equilibrium is broken. Find how motion begins.

14. Find the horizontal force which must be applied to the centre of a carriage wheel of radius r and weight W to draw it over an obstacle of height h.

Shew from the result that of wheels having the same weight, the largest is most easily drawn over a small obstacle.

15. A railway carriage whose C. G. is in the plane of the two axles rests on a rough plane, the inclination of which is supposed to be gradually increased. If either pair of wheels be locked, what is the inclination of the plane when the carriage is on the point of slipping? Which pair must be locked for the greatest inclination? There is supposed to be no friction at the axles.

16. A ladder rests against a wall. If m and n be the distances of the C. G. from its ends, and ϕ, ϕ' the angles of friction of the ground and wall respectively; find the position of the ladder when it is just about to slip.

17. A cord wound round a cylinder resting on a rough

inclined plane, is carried over a pulley and supports a weight. Find how motion begins.

18. Two weights connected by a string support each other on two inclined planes which have a common vertex. If the weights receive a small displacement, shew that their virtual velocities are inversely as the weights.

ANSWERS.

CHAPTERS I. AND II.

1. 8·784 lbs. 2. ·405 in. 3. 30 lbs.

4. $7\sqrt{3}$ lbs. 30°. 5. 7.071 lbs. 6. 150°.

7. 32·233 lbs. 19·467 lbs. 8. 11·812 lbs. 23° 7′.

9. 4·63 lbs. 10. $1 : \sqrt{3} : 2$. 11. 5 lbs.

12. (1) No. (2) Yes; the forces act in the same line, and the third is opposite the first two. (3) Yes.

14. $17\sqrt{3}$ lbs. 15. $1\frac{1}{8}$ tons.

CHAPTER III.

1. 365 lbs. $16\frac{208}{365}$ inches and $7\frac{157}{365}$ inches.

2. $121\frac{31}{35}$ lbs. and $115\frac{4}{35}$ lbs.

3. 16 lbs. The point at which resultant acts is in the bar produced on the side of the greater force, and is distant $17\frac{1}{16}$ in. from this force.

4. $\frac{1}{3}(Q+2P)$ and $\frac{1}{3}(P+2Q)$.

6. Tensions of AC and BC are 150 lbs. and 52 lbs., pressure along the rod $= 40$ lbs., resultant action on the peg $= 182$ lbs.

CHAPTERS IV. AND V.

1. $1\frac{88}{157}$ inches from the middle point.

2. $3\frac{178}{218}$ feet.

3. On the bisector of A at a distance $\frac{12}{5}\sqrt{3}$ inches from A.

4. 18·028 inches. 5. 2 inches.

6. The distance of the C. G. from the lighter end is $\frac{11}{18}$ of the length of the rod.

7. The distance of the required C. G. from the centre of the larger circle is $\frac{1}{6}$ its radius.

CHAPTERS VI. AND VII.

1. 6 inches. 2. 19 : 24. 3. $16\sqrt{\frac{2}{3}}$ lbs.

4. $W = \sqrt{(PQ)}$ 3 lbs. 5. $1\frac{11}{37}$ inches.

6. 347 feet, 485·8 feet. 7. 13 lbs. $13\sqrt{3}$ lbs.

8. $\frac{26}{3}\sqrt{3}$ lbs. 9. $8(\sqrt{3}-1)$ lbs. 10. 678·58 lbs.

11. ·396 lbs. 12. 6 cwt. 59 lbs. 13. 7.

14. 100 lbs. 15. Weight 1555 lbs., strain 1675 lbs.

16. Inclination of force to plane $= 60°$.

17. $\mu = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$.

ANSWERS, &c. TO PROBLEMS.

1. If θ be the inclination of the beam to the vertical, and the centre of gravity of the beam cut it in the ratio $m:n$, then
$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta;$$
the pressures are, $R = W.\dfrac{\sin\beta}{\sin(\alpha+\beta)}$, $R' = W\dfrac{\sin\alpha}{\sin(\alpha+\beta)}$.

2. $(m+n)\cot\theta = m\cot(\alpha+\phi) - n\cot(\beta-\phi)$.

3. If θ be the inclination of beam to the vertical
$$4r\sin^2\theta - a\sin\theta - 2r = 0.$$

4. If a = distance of C. G. from the end A, and c = distance of peg from the wall, θ = inclination of beam to wall,
$$\sin\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}}.$$

5. Take moments about the point where the reactions of the wall and the ground meet. If the surfaces be considered smooth, and α, β be the inclinations of the beam and string to the horizon, then
$$\text{tension} = \frac{W\cos\alpha}{2\sin(\alpha-\beta)}.$$

6. Use the formula $(m+n)\cot\theta = n\cot A - m\cot B$.

7. The vertical through the C. G. of the cone passes through the peg, and bisects the angle formed by the string.

9. If W be the weight of the cylinder, α, β, γ the inclinations of the planes and string to the horizon, then limits of P are $W.\dfrac{\sin\beta}{1+\cos(\beta-\gamma)}$, and $W.\dfrac{\sin\alpha}{1+\cos(2\beta-\gamma-\alpha)}$.

10. The C. G. lies on the line bisecting the parallel sides, and cuts it in the ratio of $a+2b:2a+b$.

11. $P\sin\alpha = Q\sin\beta = T.$

12. If α, β, γ, ... λ be the inclinations of the several parts of the string to the vertical $T_1 \ldots T_n$,

$$T_1\sin\alpha = T_2\sin\beta = \ldots = T_n\sin\lambda = (P+Q+R+\ldots)\frac{\sin\lambda\sin\alpha}{\sin(\lambda-\alpha)}.$$

Hence it appears that the horizontal components of the tensions of the several parts of the string are the same. This system is called the Funicular Polygon.

13. The motion is one of sliding at P, and perfect rolling at R.

14. $P = W\dfrac{\sqrt{\{h(2r-h)\}}}{r-h}.$

When h is small compared with r, $P = W\sqrt{\left(\dfrac{2h}{r}\right)}$.

15. If μ be the coefficient of friction, r the radius of each wheel, $2a$ the distance between the axles, then the inclination α of the plane to the horizon is given by

$$\cot\alpha = \frac{2}{\mu} \pm \frac{r}{a},$$

the sign being + or − according as the higher or lower pair of wheels is locked.

Hence, for the greatest inclination the lower pair of wheels must be locked.

16. If θ be the inclination of the ladder to the vertical

$$(m+n)\cot\theta = m\cot\phi - n\tan\phi'.$$